Journal of the Nigerian Association of Mathematical Physics
***Volume* 22 (November, 2012), pp 5 – 8**
© J. of NAMP

Nuclear Automorphism of a class of Osborn Loops

[1]John Olusola Adeniran and [2*]Abednego Orobosa Isere

[1]Department of Mathematics,
University of Agriculture, Abeokuta , Nigeria.
[2]Department of Mathematics,
Ambrose Alli University Ekpoma, Nigeria.

Abstract

Let $(L,\cdot)$ be any loop and $\emptyset$ an automorphism of L into L such that $x\emptyset y \cdot zx = x\emptyset(yz \cdot x)$ $\forall\ x,y,z \in L$. Then, it is shown that $A(L)$ -holomorph $(H,\circ)$ of $(L,\cdot)$ is an Osborn loop with inverse property where $A(L)$ is a group of all automorphisms of $(L,\cdot)$. Moreover, it is established that $(H,\circ)$ is an Osborn loop whenever $x^\lambda \cdot x\emptyset$ is in the nucleus of $(L,\cdot)$ and thus making each of the automorphisms of $(L,\cdot)$ a nuclear automorphism.

2000 Mathematics Subject Classification. Primary 20N05; Secondary 08A05
Keywords : Osborn loops, isotopism, autotopism, Nuclear Automorphism.

1.0 Introduction

A non-void set G on which is defined a binary operation $(\cdot)$ such that G is closed under the binary operation $(\cdot)$ is called a groupoid. Associative groupoids are known as semigroups. A semigroup, with two-sided identity such that $a \cdot e = a = e \cdot a$ for all $a \in G$, is known as a monoid. An invertible monoid is a group. On the other hand, a groupoid $(G,\cdot)$ is a quasigroup if the equation $xy = z$ has a unique solution in G whenever two of the three elements $x, y, z \in G$ are specified. A loop L is a quasigroup with a two-sided identity 'e' such that $a \cdot e = a = e \cdot a$ for all $a \in L$. All groups are loops but all loops are not groups. Those that are groups are called associative loops.

A loop $(L,\cdot)$ is called an Osborn loop if it obeys the identity:

$$(x^\lambda \backslash y) . zx = x\ (yz . x) \qquad (1)$$

for all $x, y, z \in L$. The term Osborn loops first appeared in a work of Huthnance Jr [1] in 1968, on generalized Moufang loops. However, the equation (1) is according to Basarab and Belioglo [2]. For detail review of Osborn loops-see [3 - 11]. Moreover, the most popularly known varieties of Osborn loops, are CC-loops, Moufang loops, VD-loops and universal WIPLs –see [5],[6].

However, Osborn loops are one of the least studied loops due to the nature of the loops, compare to most loops of Bol-Moufang types- see [12 - 27]. Most varieties of Osborn loops are not inverse property loops. Thus, holomorphy (nuclear automorphism) of Osborn loops is quite challenging- see Isere *et al* [8] submitted in another Journal.

Interestingly, Adeniran [23] and Robinson [26], Oyebo and Adeniran[27], and Adeniran *et al* [24], Chiboka and Solarin [17], Bruck [19], Bruck and Paige [20], Robinson [25], Huthnance [1], and Adeniran [22] have studied the holomorph of Bol loops; central loops; conjugacy closed loops; inverse property loops; A-loops; extra loops; weak inverse property loops and Osborn loops; and Bruck loops respectively.

Bruck [19] has shown that the holomorph of a loop is also a loop. Consequently, the concept of holomorphy of loops became exciting to many researchers. However, this is more interesting when a loop satisfies inverse property. Generally, most of the varieties of Osborn loops are not inverse property loops. Therefore, our results herein are applicable only to a class of Osborn loops. That is, to Osborn loops which satisfy inverse property [16].

Therefore, this paper is devoted to formulating a necessary and sufficient condition for a holomorph of an inverse property Osborn loop to be an Osborn loop. It also examines the condition for the automorphism of an Osborn loop to be nuclear. We wish to formally define these terms in section 2.

*Corresponding author: ***Abednego Orobosa Isere,*** E-mail: abednis@yahoo.co.uk, Tel. +234 8055862195

2.0 Preliminaries

Definition 2.1 A loop $(L,\cdot)$ is called a left inverse property loop (LIPL) if it has the left inverse property (LIP) i.e. if there exists a bijection: $J_\lambda: x \to x^\lambda$ on L such that $x^\lambda \cdot xy = y$ and a right inverse property loop (RIPL) if it has the right inverse property (RIP) i.e. if there exists a bijection: $J_\rho: x \to x^\rho$ on L such that $yx \cdot x^\rho = y$ for all $x, y, z \ in\ L$.

A loop that satisfies, both right and left inverse properties is simply called an inverse property loop (IPL) [9], [21].

Theorem 2.1 Huthnance [1]. The holomorph of a quasigroup (loop) L is a AIPL (AIPL) or CIPQ (CIPL) if and only if Aum(L) ={1} and L is a AIPLQ (AIPL) or CIPQ (CIPL).

Theorem 2.2 [15] Let $(L,\cdot)$ be an LC-loop (RC-loop). Then :

1. $(L,\cdot)$ is a left (right) alternative loop,
2. $(L,\cdot)$ is a left (right) inverse property loop,
3. $(L,\cdot)$ is a left (right) nuclear square loop,
4. $(L,\cdot)$ is a left (right) power alternative loop,
5. $(L,\cdot)$ is a middle square loop,
6. $(L,\cdot)$ is power associative loop.

Definition 2.2 A triple (α, β, γ) of bijections is called an isotopism of loop $(L,\cdot)$ onto a loop $(H, ^\circ)$ provided $x\alpha \circ y\beta = (x \cdot y)\gamma\ \forall\ x, y \in L. (H,\circ)$ is called an isotope of $(L,\cdot)$. The loops $(L,\cdot)$ and $(H,\circ)$ are said to be isotopic to each other. – see [9]

Definition 2.3 Let α and β be a permutation of L and let i denote identity map on L. Then (α, β, ι) is a principal isotopism of a loop $(L,\cdot)$ onto a loop $(L,\circ)$ which imply that (α, β, ι) is an isotopism of $(L,\cdot)$ onto $(L,\circ)$.

Definition 2.4 An isotopism of $(L,\cdot)$ onto $(L,\cdot)$ is called an autotopism of $(L,\cdot)$. The group of autotopism of L is denoted by $A(L)$.

Remark 2.1 The components of isotopism are usually denoted by lower case Greek letters, thus if $T = (U, V, W)$ is an autotopism of a loop $(L,\cdot)$, then

$$xU \cdot yV = (xy)W, \forall\ x, y \in L. \quad (2)$$

The set of all autotopism of a loop is a group with the inverse of $TT^{-1} = (U, V, W)^{-1} = (U^{-1}, V^{-1}, W^{-1})$. The identity element of the group being (I, I, I) where I is the identity map of L. If $T = (U, U, U)$, then T is called the automorphism of $(L,\cdot)$ [14].

Definition 2.5 Let $(L,\cdot)$ be an inverse property loop with the nucleus denoted by N. Then an automorphism α of $(L,\cdot)$ is left nuclear iff $a\alpha \cdot a^{-1} \in N$ for all $a \in L$.

Definition 2.6 Let $(L,\cdot)$ be a loop, $A(L)$ a group of automorphisms of loop $(L,\cdot)$ and let $HH = A(L) \times L$ and define

$$(\alpha, x) \circ (\beta, y) = (\alpha\beta, x\beta \cdot y) \quad (3)$$

$V(\alpha, x), (\beta, y) \in H$. Then the loop $(H,\circ)$ is called the $A(L) - holomorph$ of $(L,\cdot)$ or simply holomorph of $(L,\cdot)$.

3.0 Holomorphy of Osborn loops

Theorem 3.1

Let $(L,\cdot)$ be an inverse property loop and $A(L)$ be a group of automorphism of $(L,\cdot)$. Then $A(L)$ -holomorph $(H,\circ)$ of $(L,\cdot)$ is an Osborn loop if

$$x\emptyset y \cdot zx = x\emptyset(yz \cdot x) \quad (4)$$

such that every $\emptyset \in A(L)$ is an identity mapping.

Proof:

Suppose $A(L)$-holomorph $(H,\circ)$ of $(L,\cdot)$ is an Osborn loop we have

$$\{(\alpha, x) \circ (\beta, y)\} \circ \{(\gamma, z) \circ (\alpha, x)\} = (\alpha, x) \circ \{[(\beta, x) \circ (\gamma, z)] \circ (\alpha, x)\} \quad (5)$$

$$(\alpha\beta, x\beta \cdot y) \circ (\gamma\alpha, z\alpha \cdot x) = (\alpha, x) \circ \{(\beta\gamma, y\gamma \cdot z) \circ (\alpha, x)\} \quad (6)$$

$$(\alpha\beta \cdot \gamma\alpha, (x\beta \cdot y)\gamma\alpha \cdot z\alpha \cdot x) = (\alpha, x) \circ \{\beta\gamma \cdot \alpha, (y\gamma \cdot z)\alpha \cdot x\} \quad (7)$$

$$\{(\alpha\beta \cdot \gamma\alpha, (x\beta \cdot y)\gamma\alpha \cdot z\alpha \cdot x)\} = \{\alpha(\beta\gamma \cdot \alpha), x(\beta\gamma \cdot \alpha) \cdot [(y\gamma \cdot z)\alpha \cdot x]\} \quad (8)$$

$\forall\ x, y, z \in L$ and $\alpha, \beta, \gamma \in A(L)$. Therefore,

$$(x\beta \cdot y)\gamma\alpha \cdot z\alpha \cdot x = x(\beta\gamma \cdot \alpha) \cdot \{(y\gamma \cdot z)\alpha \cdot x\} \quad (9)$$

$$(x\beta \cdot y)\gamma\alpha \cdot z\alpha \cdot x = x\beta \cdot \gamma\alpha \cdot [y\gamma \cdot z\alpha \cdot x] \quad (10)$$

Put $\emptyset = \gamma\alpha, z\alpha = \bar{z}$ such that $y\gamma = y$ and $x\beta = x$

$$xy\emptyset \cdot zx = x\emptyset(yz \cdot x) = x\emptyset y \cdot zx = x\emptyset(yz \cdot x) \text{ since } \emptyset = I, \quad (11)$$

It is obvious that if $\emptyset = I$, we have $xy \cdot zx = x(yz \cdot x)$ which is an Osborn identity.

Corollary 3.1

Let $(L,\cdot)$ be a loop, and $A(L)$ be the group of all automorphism of L whose elements are identity mapping, then L is an Osborn loop if and only if

$$\mathcal{A} = \langle R_{\emptyset}L_x, R_x, R_xL_{x\emptyset}\rangle \quad (12)$$

is an autotopism of $L, \forall\, x \in L$ and $\emptyset \in A(L)$.

Proof:

This is a consequence of the above theorem.

i.e. applying the autotopism to the product yz we have

$$yR_{\emptyset}L_x \cdot zR_x = yzR_xL_{x\emptyset} = xy\emptyset \cdot zx = (yz \cdot x)L_{x\emptyset} \quad (13)$$

$$xy\emptyset \cdot zx = x\emptyset(yz \cdot x) = x\emptyset y \cdot zx = x\emptyset(yz \cdot x) \text{ since } \emptyset = I \quad (14)$$

It is obvious that if $\emptyset = I$, we have

$$xy \cdot zx = x(yz \cdot x) \quad (15)$$

The converse follows from theorem 3.1

4.0 Nuclear Automorphism

Theorem 4.1

If $(L,\cdot)$ is an Osborn loop with the inverse property and $A(L)$ a group of automorphism of $(L,\cdot)$. Then the $A(L)$ -holomorph $(H,\circ)$ of $(L,\cdot)$ is an Osborn loop if $(L,\cdot)$ is an Osborn and each $\emptyset \in A(L)$ is a left nuclear automorphism of $(L,\cdot)$.

Proof:

Suppose $(H,\circ)$ is an Osborn. Since $(L,\cdot)$ is isomorphic to a subloop of $(H,\circ)$, it follows that $(L,\cdot)$ must be an Osborn loop. From equation (1) we have that

$$A_\lambda(x) = \langle L^{-1}_{x^\lambda}, R_x, R_xL_x\rangle \;\; \forall\, x \in L \quad (16)$$

and

$$B_\lambda(x) = \langle L_x, R_x, R_xL_x\rangle \;\; \forall\, x \in L \quad (17)$$

$B^{-1}_\lambda(x) = \langle L^{-1}_x, R^{-1}_x, R^{-1}_xL^{-1}_x\rangle \;\forall\, x \in L$ is also an autotopism of L .

Recall

$x\emptyset y \cdot zx = x\emptyset(yz \cdot x) \;\; \forall\, x, y, z \in L \textit{ and } \emptyset \in A(L)$.

$\Rightarrow \langle L_{x\emptyset}, R_x, R_xL_{x\emptyset}\rangle = P_\lambda(x)$ is an autotopism

$$P_\lambda B^{-1}_\lambda = \langle L_{x\emptyset}, R_x, R_xL_{x\emptyset}\rangle\langle L^{-1}_x, R^{-1}_x, R^{-1}_xL^{-1}_x\rangle \quad (18)$$

$$= \langle L_{x\phi}L_{x^\lambda}, I, R_xL_{x\emptyset}R^{-1}_xL^{-1}_x\rangle \quad (19)$$

as an autotopism of $L, \forall\, x \in L$ and $\emptyset \in A(L)$

then applying (19) to $1 \cdot b$ we have

$$1L_{x\emptyset}L_{x^\lambda} \cdot b = (1 \cdot b)R_xL_{x\emptyset}R^{-1}_xL^{-1}_x \quad (20)$$

$$\left(x^\lambda \cdot x\emptyset\right)b = bR_xL_{x\emptyset}R^{-1}_xL^{-1}_x \quad (21)$$

$$bL_{(x^\lambda \cdot x\emptyset)} = bR_xL_{x\emptyset}R^{-1}_xL^{-1}_x \quad (22)$$

Therefore,

$$L_{(x^\lambda \cdot x\emptyset)} = R_xL_{x\emptyset}R^{-1}_xL^{-1}_x \quad (23)$$

$\forall\, x \in L$ and $\emptyset \in A(L)$. If we put equation (23) into (19), we have

$$P_\lambda(x)B^{-1}_\lambda(x) = \langle\, L_{(x^\lambda \cdot x\emptyset)}, I,\; L_{(x^\lambda \cdot x\emptyset)}\rangle \quad (24)$$

$\forall\, x, x^\lambda \in L$ and $\emptyset \in A(L)$. Consequently $(x^\lambda \cdot x\emptyset) \in N_\lambda(L)$ and hence $\emptyset \in A(L)$, is left nuclear.

Conclusion

Since $(L,\cdot)$ is an Osborn loop which satisfies inverse property, then the nuclei are equal. Thus $\left(x^\lambda \cdot x\emptyset\right) \in N(L)$, consequently, $\emptyset \in A(L)$ is a left nuclear automorphism. Also by using this definition of Osborn loops: $xy \cdot (z\,/\,x^\rho) = (x \cdot yz)x$ and following the same procedure one can show that $(x\emptyset \cdot x^\rho) \in N_\rho(L)$, and hence $(x\emptyset \cdot x^\rho) \in N(L)$, and $\emptyset \in A(L)$ is right nuclear automorphism.

Acknowledgement

The second author wishes to express his profound gratitude to the management of Education Trust Fund Academic Staff Training and Development-2009 (ETF AST & D) for the grant given him to carry out this research and also to the management of Ambrose Alli University for their joint support of the grant.

References

[1] E.D. Huthnance Jr. (1968), A theory of generalized Monfang Loops, Ph.D. Thesis, Georgia institute of technology, Georgia.
[2] A.S Basarab and A.I Belioglo (1979), Universal automorphic inverse G-loop, Quasi-groups and loops, Mat. Issled. 71, 3-7
[3] J.M. Osborn (1959), A theorem on A-loops, Proc. Amer. Math. Soc 9, 347-349.
[4] J.M Osborn (1961), Loops with the week inverse property, Proc. J. Math. 10, 295-304.
[5] T.G Jaiyeola (2008), The study of the University of Osborn Loops, Ph.D thesis, university of Agriculture, Abeokuta.
[6] T.G. Jaiyeola and Adeniran (2000), New identities in universal Osborn Loops quasi groups and related systems, moldova 17(1)
[7] A.O. Isere, J.O Adeniran and A.R.T Solarin (2012), Some examples of finite Osborn loops, accepted in JNMS.
[8] A.O Isere, J.O Adeniran and T.G Jaiyeola, Holomorphy of Osborn loops- sent out for publication in the Canadian Journal on Science and Engineering.
[9] H.O. Pflugfelder (1990), Qusaigroups and Loops : Introduction, Sigma Series in Pure Math. 7, Heldermann verlag, Berlin, , 147pp.
[10] M.K. Kinyon, K. Kunen, J.D. Phillips (2002), A generalization of Moufang and Steiner loops, Alg. Univer. 48,1,81-101.
[11] M.K. Kinyon (2005), A survey of Osborn loops Milehigh conference on loops, quasi-groups and non-associative systems, university of Denver, Denver, Colorado.
[12] K. Kunen (1996a), Quasigroups, Loops and Associative Laws, J. Alg. 185, 194-204.
[13] K. Kunen (1996b), Moufang Quasigroups, J. Alg. 183, 231-234.
[14] J.D. Phillips and P. Vojtechovsky(2005a), The varieties of loops of Bol-Moufang type, Alg. Univ., 53(3), 115-137.
[15] J.D. Phillips and P. Vojtechovsky (2005b), The varieties of quasigroups of Bol-Moufang type: An equational reasoning approach, J. Alg.,293, 17-33.
[16] V.O. Chiboka (1990), The Study of properties and Construction of Certain Finite Order G-loops., Ph.d Thesis, Obafemi Awolowo University, Ile-Ife, Nigeria.
[17] V.O. Chiboka and A.R.T. Solarin, Holomorphs of conjugacy closed Loops (1991)., Scientific Annals of ‘’AL.I.CUZA’’., 38, 277-283.
[18] V.O. Chiboka and A.R.T. Solarin (1993), Autotopism Characterization of G-loops, Scientific Annals of ‘’AL.I.CUZA’’., 40, 19-26.
[19] R.H. Bruck (1944), Contribution to the theory of Loops., Trans. Amer. Math. Soc., 55, 245-354.
[20] R.H Bruck and L.J. Paige (1956), Loops whose inner mappings are automorphisms, the annuals of mathematics, 63,2, 308-323
[21] R.H. Bruck (1966), A Survey of Binary Systems. Springer-Verlag, Berlin-Gottinge-Heidelbery.
[22] J.O. Adeniran (2004), On Some Maps of Conjugacy Closed Loops. An. Stiint. Univ. ‘’AL.I.Cuza’’. Iasi. Mat., 51, pp. 267-272
[23] J.O Adeniran (2005), On Holomorphic theory of a class of left Bol Loops, AJ.I. Cuza 51,1 23-28
[24] J.O Adeniran, Y. T. Oyebo and D. Mohammed (2011), On certain isotopic maps of central loops, Proyeccions 30, 303-318
[25] D.A. Robinson (1971), Holomorphy Theory of Extra Loops., Publ. Math. Debrecen., 18, 59-64.
[26] D.A. Robinson (1977), A special Embedding of Bol-Loops in Groups., Acta Math. Hungaricae., 18, 95-113.
[27] Y.T. Oyebo and J.O. Adeniran(1996), On the Holomoph of Central Loops, Pre-print.