# Kronecker Product of Non-Reduced Burau Representation of The Braid Group, $\mathbf{B}_{3}$. <br> ${ }^{1}$ Fadipe-Joseph, Olubunmi A., ${ }^{2 *}$ Tele, Fwangshak M. and ${ }^{3}$ Makanjuola, Samuel O. <br> Department of Mathematics, University of Ilorin, P.M.B. 1515, Ilorin, Nigeria. 


#### Abstract

In this paper, we construct the kronecker product of complex specializations of the non-reduced Burau representation of the braid group, $B_{3}$. It turns out that, although the images are irreducible representations, the tensored mapping does not define a representation of $B_{3}$.


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### 1.0 Introduction

The braid group of n strings, $B_{n}$, is an abstract group which has a presentation with generators $\sigma_{1}, \sigma_{2, \ldots,}, \sigma_{n-1}$, and defined by the relations

$$
\left\{\begin{array}{c}
\sigma_{i} \sigma_{i+1} \sigma_{i}=\sigma_{i+1} \sigma_{i} \sigma_{i+1}, \text { for } i=1,2, \ldots, n-2  \tag{1}\\
\sigma_{j}=\sigma_{j} \sigma_{i}, \text { if }|i-j| \geq 2
\end{array}\right.
$$

The generators $\sigma_{1}, \sigma_{2, \ldots, \ldots} \sigma_{n-1}$ are called standard generators.
Since the discovery of the braid group in 1926 by Artin, concerted efforts have been made by researchers in providing linear representations for the braid group. Such representations include Burau representation, Gassner representation, Jones representation, Lawrence krammer representation, Wada's representation and so on. Perhaps, the most significant of these is the Burau representation which dates back to 1935. (see Birman [1]).

Generally, group representation is concerned mainly with the use of algebras like modules, permutations and matrices in place of the elements of an abstract group or, less generally, of a group arising from geometric symmetries. A matrix representation of a group $G$ is a homomorphism, $T$ of $G$ into a subgroup of $G L_{n}(\mathbb{C})$, where the integer $n$ is the degree or dimension of $T$. The nonreduced Burau representation is actually a matrix representation which offers useful information about the braid group.

Abdulrahim and Zeid [2] studied the irreducibility of the tensor ( kronecker) product of specializations of Burau representation of the braid group. It was shown that the tensor product of an irreducible $\beta_{n}(y)$ or $\hat{\beta}_{n}(y)$ with an irreducible $\beta_{n}(z)$ or $\hat{\beta}_{n}(z)$ is irreducible if and only if $y \neq z^{ \pm 1}$. Indeed, this result was a generalization of the one earlier obtained in [3].

Another important representation of the braid group is the Wada's representation which came to lime light in 1992. Recently, Zeid and Abdulrahim [4] introduced the operation of tensor product on the Wada's representation of $B_{3}$. The construction started by considering
$\beta_{n}(x) \otimes \beta_{n}(y): B_{3} \rightarrow G L\left(\mathbb{C}^{3} \otimes \mathbb{C}^{3}\right)$, where $\beta_{n}(x): B_{3} \rightarrow G L\left(\mathbb{C}^{3}\right)$ and $\beta_{n}(y): B_{3} \rightarrow G L\left(\mathbb{C}^{3}\right)$ are two irreducible Wada's representation of the braid group; and $x$ and $y$ are nonzero complex numbers used in defining Wada's representation.
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For the purpose of this paper, we shall consider, analogously, the tensor product of nonreduced Burau representation of the braid group of three strings, $B_{3}$. It is our interest to answer the question: If $\gamma_{n}(x): B_{3} \rightarrow G L\left(\mathbb{C}^{3}\right)$ and $\gamma_{n}(y): B_{3} \rightarrow G L\left(\mathbb{C}^{3}\right)$ are two irreducible nonreduced Burau representations of $B_{3}$, with $x, y \in \mathbb{C}^{*}$, does the tensor product
$\gamma_{n}(x) \otimes \gamma_{n}(y): B_{3} \rightarrow G L\left(\mathbb{C}^{3} \otimes \mathbb{C}^{3}\right)$ of complex specializations of the nonreduced Burau representation of $B_{3}$ give another representation of $B_{3}$ ? From there, we make a salient remark to conclude the work.

### 2.0 Preliminaries\Theoretical Framework

Definition 1. Burau Representation
The (non-reduced) Burau representation $\beta_{n}(t): B_{3} \rightarrow G L_{n}\left(\mathbb{C}\left[t^{ \pm 1}\right]\right)$ is defined by
$\beta_{n}(t):\left(\sigma_{i}\right)=\left(\begin{array}{cccc}I_{i-1} & 0 & 0 & 0 \\ 0 & 1-t & t & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & I_{n-i-1}\end{array}\right)$ for $i=1,2, \ldots, n-1$
where $I_{k}$ denotes $k \times k$ identity matrix, $t$ is an indeterminate and $\mathbb{C}\left[t^{ \pm 1}\right]$ is a Laurent polynomial ring over the complex numbers.
Specializing $t \rightarrow x$, where $x \in \mathbb{C}^{*}$, defines a representation
$\beta_{3}(x): B_{3} \rightarrow G L_{3}(\mathbb{C})$, which we call irreducible complex specialization of the braid group, $B_{3}$.
Definition 2. Equivalent Representation[5]
Let $A(x)$ and $B(x)$ be $n$-dimensional representations of a group $G$. We say that $A(x)$ and $B(x)$ are equivalent if there exist a non-singular matrix $T$ over $K$ (a scalar field) such that $B(x)=T^{-1} A(x) T$ for all $x \in G$.

Definition 3. Reducible Representation [5]
The matrix representation $A(x)$ is reducible over $K$ if there exist a nonsingular matrix $T$ over $K$ such that $B(x)=T^{-1} A(x) T$ for all $x \in G$.

$$
B(x)=\left(\begin{array}{cc}
C(x) & \overline{0} \\
E(x) & D(x)
\end{array}\right)
$$

where if $A(x) \in G L_{m}(K), C(x)$ an $r \times r, D(x)$ an $s \times s$ matrices, then $E(x)$ is an $s \times r$ matrix and $m=s+r$.
Equivalently, for a group $G$ with some generators, a representation of $G \longrightarrow G L_{m}(\mathbb{C})$ is reducible if there exists a nonzero proper subspace of $\mathbb{C}^{n}$ that is invariant under the action of the generators. If a representation is not reducible, it is said to be irreducible.
Theorem 1. Let $A(x)$ be a matrix representation of $G$ of degree $m$ over $K$. Then either $A(x)$ is irreducible or else

$$
\left(\begin{array}{ccccc}
A_{1}(x) & 0 & 0 & \ldots & 0  \tag{2}\\
A_{21}(x) & A_{2}(x) & 0 & \ldots & 0 \\
A_{31}(x) & A_{32}(x) & A_{3}(x) & \ldots & 0 \\
\vdots & \vdots & \vdots & & \vdots \\
A_{l 1}(x) & A_{l 2}(x) & A_{l 3}(x) & \ldots & A_{l}(x)
\end{array}\right) \sim A(x)
$$

where $\sim$ denotes equivalence and $A_{1}(x), A_{2}(x), \ldots, A_{l}(x)$ are irreducible representations over $K$.
The theorem states that every matrix representation can be brought into lower triangular block form in which the diagonal blocks are irreducible.

Definition 4. Kronecker Product
The kronecker product (or matrix direct product or tensor product) of a $p \times q$ matrix $A$ and an $m \times n$ matrix $B$ is denoted by $A \otimes B$ and is a $(p m) \times(q n)$ matrix defined by

$$
A \otimes B=\left(\begin{array}{cccc}
A_{11} B & A_{12} B & \ldots & A_{1 q} B  \tag{3}\\
A_{21} B & A_{22} B & \ldots & A_{2 q} B \\
\vdots & \vdots & & \vdots \\
A_{p 1} B & A_{p 2} B & \ldots & A_{p q} B
\end{array}\right)
$$

This definition is, in fact, the left kronecker product, to which we adhere in this paper strictly. Note in general that $A \otimes B \neq$ $B \otimes A$. The properties of kronecker product can be found in [6].

### 3.0 Main Result

Theorem 2 The complex specializations of nonreduced Burau representation of $B_{3}$ are irreducible, but their kronecker product does not define a representation of $B_{3}$.
Proof: Let $x$ and $y$ be nonzero complex numbers. By specializing $t \rightarrow x$ in the nonreduced Burau representation (where $\beta_{n}(t)$ is now $\gamma_{n}(t)$ in definition 1 , the generating images of $B_{3}$ under the mapping $\gamma_{3}(x): B_{3} \rightarrow G L_{3}(\mathbb{C})$ are given by

$$
\rho_{1}=\left(\begin{array}{ccc}
1-x & x & 0 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{array}\right) \quad \text { and } \quad \rho_{2}=\left(\begin{array}{ccc}
1 & 0 & 0 \\
0 & 1-x & x \\
0 & 1 & 0
\end{array}\right)
$$

Similarly, the generating images of $B_{3}$ under the mapping
$\gamma_{3}(y): B_{3} \longrightarrow G L_{3}(\mathbb{C})$ are given by

$$
\rho_{1}=\left(\begin{array}{ccc}
1-y & y & 0 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{array}\right) \quad \text { and } \quad \rho_{2}=\left(\begin{array}{ccc}
1 & 0 & 0 \\
0 & 1-y & y \\
0 & 1 & 0
\end{array}\right)
$$

The irreducibility of these specializations is evident from definition 3 and Theorem 2.1. It remains to show that their tensor product is not a representation of $B_{3}$.
To do this, we take the kronecker product of these specializations according to definition 4, and then we obtain the mapping $\gamma_{n}(x) \otimes \gamma_{n}(y): B_{3} \rightarrow G L\left(\mathbb{C}^{3} \otimes \mathbb{C}^{3}\right)$ whose images are $9 \times 9$ matrix as follows

$$
\rho_{1}=\left(\begin{array}{ccccccccc}
(1-x)(1-y) & (1-x) y & 0 & x(1-y) & x y & 0 & 0 & 0 & 0  \tag{4}\\
1-x & 0 & 0 & x & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1-x & 0 & 0 & x & 0 & 0 & 0 \\
1-y & y & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1-y & y & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right)
$$

and

$$
\rho_{2}=\left(\begin{array}{ccccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0  \tag{5}\\
0 & 1-y & y & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1-x & 0 & 0 & x & 0 & 0 \\
0 & 0 & 0 & 0 & (1-x)(1-y) & (1-x) y & 0 & x(1-y) & x y \\
0 & 0 & 0 & 0 & 1-x & 0 & 0 & x & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1-y & y & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0
\end{array}\right)
$$

That these images do not define a representation of the braid group $B_{3}$ is obvious.
This is done by verifying the relations (1) that define the braid group $B_{3}$. The verification shows that $\rho_{1} \rho_{2} \rho_{1} \neq \rho_{2} \rho_{1} \rho_{2}$. Moreover, Theorem 2.1 shows that this kind of matrices cannot be a representation for such a group.

### 4.0 Conclusion

Bases on the result obtained in this paper, it is safe to conclude that, another representation for the braid group, $B_{3}$ cannot be obtained by tensoring specializations of 3-dimensional irreducible (non-reduced) Burau representations of $B_{3}$.

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