General Relativistic Theory of Radar Sounding in the Gravitational Field of an Oblate Spheriodal Sun to the Order of C⁻³

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Abstract

The resolution of radar sounding in the gravitational field exterior to the spherical massive sun to the order of C^{-3} and C^{-5} is well known. In this paper an attempt is made to resolve the radar sounding phenomenon in the gravitational field exterior to a spheriodal massive sun to the order of C^{-3}. This is important because the shape of the sun and other terrestrial bodies are spheriodal in nature and so the usual assumption of the spherical shape is purely an assumption.

Keywords: Radar Sounding Oblateness of the Sun

1.0   Introduction

Since 1915 when Einstein came out with his gravitational field equation, which is a second order non-linear partial differential equation, only two exact solutions have been provided. One by K.Schwartschild and the other by Robertson – Walker. The Schwartschild's metric has resolved satisfactorily the problems of (i) Orbital perihelion procession (ii) gravitational deflection of starlight and (iv) Radar sounding while the Robertson – Walker metric has given a lot of insight into cosmological studies.

Indeed the theory of relativity has also been applied to the radar – sounding phenomenon to the order of c^{-3} (Jabil and Howusu, 2002), and recently extended to c^{-5}(Jabil and Bakwa 2008). In developing the field equations for solving the above mentioned physical phenomenon, the theory of relativity considered the massive sun, planetary bodies and other stars as homogenous spherical bodies. But it is well known that the only reason for these restrictions is mathematical convenience and simplicity(Howusu and Uduh 2003). The fact of nature is that the sun, which is a G2, star in the Milky Way galaxy is spheriodally oblate in shape.

Several studies and observations have been undertaken since 1966 to evaluate the solar oblateness, measurements began with the Princeton Solar Distortion Telescope and Dicke and Goldenberg (in 1967) found a value for the oblateness of the sun:

$$\varepsilon = (4.51 \pm 0.4) \times 10^{-5}$$

Goldrich and Schubert in 1968 showed that the theoretical maximum solar oblateness consistent with the stability of the sun is

$$\varepsilon = [3.5 \times 10^{-1/2}]^{13}$$

Maier, Twiyg and Sofia gave in 1992, their preliminary results of the solar diameter from a ballon flight of the Solar Disk Sextant (SDS) experiment and found to be

$$\varepsilon = 5.6 \pm 0.35 \times 10^{-5}$$

For the solar oblateness. The equatorial bulging (i.e oblateness) of the sun was also advanced by Carlip in 1996, which is an update of Micheal Weiss's work on mercury orbital precession, General Relativity and the Solar Buge. The solar Heliospherica Observatory (SₒHO) board computed the difference between solar equatorial radii and polar radii associated with the static oblateness of the sun as 8.07 ± 0.58 milliarsec (Howusu, 2005)

The Art History Club of the sun in a masterpiece gallery gave the oblateness of the sun as:

$$\varepsilon = 9.0 \times 10^{-6}$$

(http://www.myastrology.com/glossary 2000)

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The oblateness of the planets have also been determined and given as:

$$\varepsilon_m = 0.0 \qquad \text{oblateness of Mercury}$$
$$\varepsilon_v = 0.0 \qquad \text{oblateness of venus}$$
$$\varepsilon_e = 0.0034 \qquad \text{oblateness of Earth}$$
$$\varepsilon_a = 0.006 \qquad \text{oblateness of Mars}$$
$$\varepsilon_j = 0.065 \qquad \text{oblateness of Jupiter}$$
$$\varepsilon_s = 0.108 \qquad \text{oblateness of Saturn}$$
$$\varepsilon_u = 0.03 \qquad \text{oblateness of Uranus}$$
$$\varepsilon_n = 0.026 \qquad \text{oblateness of Neptune}$$

(http://hyperphysics.phy-astr.gsu.edu/hbase/solar/soldata3.html,2005).

Consequently, spheriodal geometry will have effects in the motions of all particles in the gravitational fields of the astronomical bodies.

2.0 Spheriodal Coordinates and the Metric Tensors

The oblate spheriodal coordinates of space, $(\eta, \xi, \emptyset)$ are defined in terms of the Cartesian coordinates (x, y, z) as:

$$\left. \begin{array}{l} x = a(1 - 1 - \eta^2)^{\frac{1}{2}}(1 + \xi^2)^{\frac{1}{2}} \\ \qquad y = a(1 - \eta^2)^{\frac{1}{2}}(1 + \xi^2)^{\frac{1}{2}} \\ \qquad z = a\eta\xi \end{array} \right\} \tag{1}$$

Where a is a constant parameter of a particular oblate body and
$$-1 \le \eta \le 1, \qquad 0 \le \xi < \infty, \qquad 0 \le \emptyset \le 2\pi$$

The spherical polar coordinates are related to the Cartesian coordinates (x, y, z) as:

$$\left. \begin{array}{l} x = r \sin\theta \cos\emptyset \\ y = r \sin\theta \sin\emptyset \\ z = r \cos\theta \end{array} \right\} \tag{2}$$

and inversely

$$\left. \begin{array}{l} r = (x^2 + y^2 + z^2)^{\frac{1}{2}} \\ \theta = \cos^{-1}\left\{ \dfrac{z}{(x^2+y^2+z^2)^{\frac{1}{2}}} \right\} \\ \emptyset = \tan^{-1}\left(\dfrac{y}{x}\right) \end{array} \right\} \tag{3}$$

Substituting equations (1) into equations (3), we may write

$$r(\eta, \xi, \emptyset) = a(! + \xi^2 - \eta^2)^{\frac{1}{2}} \tag{4}$$

and hence

$$dr = a(! + \xi^2 - \eta^2)^{\frac{1}{2}}(\xi d\xi - \eta d\eta) \tag{5}$$

Similarly

$$\theta(\eta, \xi, \emptyset) = \cos^{-1}\left\{ \frac{\eta\xi}{(1+\xi^2-\eta^2)^{\frac{1}{2}}} \right\} \tag{6}$$

and hence

$$d\theta = \frac{\delta\theta}{\delta\eta} d\eta + \frac{\delta\theta}{\delta\xi} d\xi \tag{7}$$

Where

$$\frac{\delta\theta}{\delta\eta} = -\frac{\xi(1+\xi^2)^{\frac{1}{2}}}{(1+\eta^2)^{\frac{1}{2}}(1+\xi^2-\eta^2)} \tag{8}$$

and

$$\frac{\delta\theta}{\delta\xi} = -\frac{(1+\eta^2)^{\frac{1}{2}}}{(1+\xi^2)^{\frac{1}{2}}(1+\xi^2-\eta^2)} \tag{9}$$

Similarly

$$\sin^2\theta(\eta, \xi, \emptyset) = \frac{(1-\eta^2 1+\xi^2)}{(1+\xi^2-\eta^2)} \tag{10}$$

The world line element in general relativity is written in terms of the metrics in spherical polar coordinates as:

$$dS^2 = g_{\mu\nu}dx^\mu dx^\nu \tag{11}$$

or

$$c^2 d\tau^2 = (1 + f[r,\theta,\emptyset])c^2 d\tau^2 (1 + f[r,\theta,\emptyset])^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta d\emptyset^2) \tag{12}$$

Where τ is proper time

t is the coordinate time

C is the speed of light vacuum

Here we have the metrics from equation (12) as

$$\left.\begin{aligned}
g_{00}(r,\theta,\emptyset) &= [1 + f(r,\theta,\emptyset)]c^2 \\
g_{11}(r,\theta,\emptyset) &= [1 + f(r,\theta,\emptyset)]^{-1} \\
g_{22}(r,\theta,\emptyset) &= -r^2 \\
g_{33}(r,\theta,\emptyset) &= r^2 sin\,^2\theta \\
g_{\mu v}(r,\theta,\emptyset) &= 0, \mu \neq v
\end{aligned}\right\} \tag{13}$$

where

$$f(r,\theta,\emptyset) = \frac{2GM}{c^2} = \frac{2}{c^2}\Phi_{sp}^+(\theta,r,\emptyset) \tag{14}$$

and

$$\Phi_{sp}^+(r,\theta,\emptyset) = -\frac{GM}{r} \tag{15}$$

Is the gravitational scalar potential exterior to a spherical massive body

M is mass of graviting body

r is radial distance

G is the universal gravitational constant

Indeed using equation (14) we may write equation (12) as:

$$c^2 d\tau^2 = c^2\left[1 + \frac{2}{c^2}\Phi_{sp}^+(r,\theta,\emptyset)\right]dt^2 - \left[1 + \frac{2}{c^2}\Phi_{sp}^+(r,\theta,\emptyset)\right]^{-1}dr^2 \tag{16}$$
$$r^2 d\theta^2 - r^2 sin^2\theta d\emptyset^2$$

Now we are interested in transforming the metrics in spherical polar coordinates to spheriodal coordinate system.

$$(r,\theta,\emptyset) \rightarrow (\eta,\xi,\emptyset)$$

and

$$\Phi_{sp}^+(r,\theta,\emptyset) \rightarrow \Phi_{ob}^+(\eta,\xi,\emptyset)$$

The gravitational scalar potential, $\Phi_{ob}^+(\eta,\xi,\emptyset)$ exterior to an oblate massive body has been derived[12] as:

$$\Phi_{ob}^+(\eta,\xi,\emptyset) = \frac{B_0P_0(\eta)Q_0(-i\xi) + B_2P_2(\eta)Q_2(-i\xi)}{P_0(\eta)Q_0} \tag{17}$$

Where Q_0 and Q_2 are legendre functions linearly independent to the legendre polynomials P_0 and P_2 respectively and B_0 and B_2 are constants.

Imposing equation (17) on equation (16) and substituting for the various parameters derived in equations (4) – (10) the metrics for the spheriodal coordinate system in terms of the gravitational scalar potential $\Phi_{ob}^+(\eta,\xi,\emptyset)$ turns out as follows:

$$g_{00}(\eta,\xi,\emptyset) + \left[1 + \frac{2}{c^2}\Phi_{ob}^+(\eta,\xi,\emptyset)\right] \tag{18}$$

$$g_{11}(\eta,\xi,\emptyset) = \frac{a^2}{(1+\xi^2-\eta^2)}\left\{\frac{\xi^2(1+\xi^2)}{(1-\eta^2)} + \eta^2\left[1 + \frac{2}{c^2}\Phi_{ob}^+(\eta,\xi,\emptyset)\right]^{-1}\right\} \tag{19}$$

$$g_{12}(\eta,\xi,\emptyset) = g_{12}(\eta,\xi,\emptyset)\frac{a\eta\xi}{(1+\xi^2-\eta^2)}\left\{1 - \left[1\frac{2}{c^2}\Phi_{ob}^+(\eta,\xi,\emptyset)\right]^{-1}\right\} \tag{20}$$

$$g_{22}(\eta,\xi,\emptyset) = \frac{a^2}{(1+\xi^2-\eta^2)}\left\{\frac{\eta^2(1+\eta^2)}{(1-\xi^2)} + \xi^2\left[1 + \frac{2}{c^2}\Phi_{ob}^+(\eta,\xi,\emptyset)\right]^{-1}\right\} \tag{21}$$

$$g_{33}(\eta,\xi,\emptyset) = -a^2(1+\xi^2)(1-\eta^2) \tag{22}$$

$$g_{\mu v}(\eta,\xi,\emptyset) = 0, \mu \neq v, otherwise \tag{23}$$

Equation (18) – (23) are the metrics in spheriodal system $(\eta,\xi,\emptyset)$

Application To Radar Sounding

Consider an observer at position r_1 sending radar signals or pulses in a radial direction toward a small body at position r_2 within the gravitational field established by the homogenous oblate spheriodal massive sun of mass M such that $r_1 > r_2$ as shown in Figure 1.
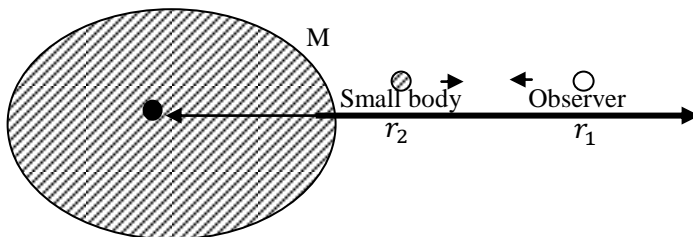

Figure 1: **"Radar Sounding Experiment"**

We are interested in computing the total time needed for the radar pulses to travel from r_1 to r_2 and back to r_1 in the radial direction within the gravitational field created by the stationary homogenous oblate sun.
Now introducing the world line element from equation (11), which may be written in the form:

$$c^2 d\tau^2 = c^2 g_{11} dt^2 + c^2 g_{11} d\eta^2 + c^2 g_{22} d\xi^2 + c^2 g_{33} d\emptyset^2 \tag{24}$$

For radial motion of a particle in the equatorial zone.

$$d\tau = 0 = d\emptyset = \eta \tag{25}$$

Hence equation (24) becomes

$$0 = c^2 g_{00} dt^2 + c^2 g_{22} d\xi^2 \tag{26}$$

Therefore the coordinate time of the radar signal is obtain from equation (26) as:

$$dt = \pm - \frac{1}{c}\left[\frac{g_{00}}{g_{22}}\right]^{-\frac{1}{2}} d\xi \tag{27}$$

The total coordinate time for round trip of the radar signals is given as:

$$Dt = -\frac{2}{c}\int_{\xi_2}^{\xi_1}\left[\frac{g_{00}}{g_{22}}\right]^{-\frac{1}{2}} d\xi + \frac{1}{c}\int_{\xi_2}^{\xi_1}\left[\frac{g_{00}}{g_{22}}\right]^{-\frac{1}{2}} d\xi \tag{28}$$

The relationship between the proper time, $D\tau$ and the coordinate time, Dt is given by:

$$D\tau = \frac{2}{c}\int_{\xi_2}^{\xi_1}\left[\frac{g_{00}}{g_{22}}\right]^{-\frac{1}{2}} d\xi \tag{29}$$

Substituting equation (29) into equation (30) we have:

$$D\tau = \left(1 - \frac{2k}{\xi_1}\right)^{\frac{1}{2}} Dt \tag{30}$$

$$D\tau = \frac{2}{c}\left(1 - \frac{2k}{\xi_1}\right)^{\frac{1}{2}}\int_{\xi_2}^{\xi_1}\left[\frac{g_{00}}{g_{22}}\right]^{-\frac{1}{2}} d\xi \tag{31}$$

Substituting (18) and (21) into equation (31) and considering that for radial motion of the signals in the direction, $\eta = 0$, and the legendre polynomials

$$P_0(\eta) = 1$$
$$P_2(\eta) = \frac{1}{2}(5\eta^2 - 3\eta)$$

Equation (31) becomes:

$$D\tau = \frac{2}{a^2 c}\left(1 - \frac{2k}{\xi_1}\right)^{\frac{1}{2}}\int_{\xi_2}^{\xi_1} 1 + \xi^{-2}\left[1 + \frac{2}{c^2}B_0 Q_0(-i\xi)\right]^{-1} d\xi \tag{32}$$

Integrating and expanding equation (32) we have:

$$D\tau = \frac{2}{a^2 c}\left\{\begin{array}{l}(\xi_1 - \xi_2)\frac{-k}{\xi_1}\left[\xi_2 - \xi_2 + \frac{1}{3}\left(\frac{1}{(\xi_1 - \xi_2)^3}\right)\right]\\ + \frac{1}{3(\xi_2 - \xi_2)^3} + \frac{B_0 Q_0(-i\xi)}{c^2}\left(\frac{1}{2(\xi_1 - \xi_2)^4}\right) + ...\end{array}\right\} \tag{33}$$

Equation (33) is the total time to the order of c^{-3} taken for the round trip of the radar signals within the gravitational field established by the homogenous spheroidal oblate sun. the expression for the radial distance, r in the equatorial plane from equation (1) in terms of ξ is given by:

$$r = a(1 + \xi^2)^{\frac{1}{2}} \tag{34}$$

and hence

$$\xi_1 = \left(\frac{r_1^2}{a^2} - 1\right)^{\frac{1}{2}}$$

and

$$\xi_2 = \left(\frac{r_2^2}{a^2} - 1\right)^{\frac{1}{2}} \tag{35}$$

Substituting equation (35) into equation (33) we have:

$$D\tau = \frac{2}{a^2c} \left\{ \begin{array}{l} \frac{1}{2a^2}(r_1^2 - r_2^2) + k\left(1 + \frac{r_1^2}{2a^2}\right)^{-1}\left[\frac{1}{2a^2}(r_1^2 - r_2^2)\right[\\ + \frac{1}{6a^2}(r_1^2 - r_2^2)^{-3}] + \frac{1}{6a^2}(r_1^2 - r_2^2)^{-3} \\ + \frac{B_0 Q_0(-i\xi)}{2a^2c^2}(r_1^2 - r_2^2)^{-4} + \cdots \end{array} \right\} \qquad (36)$$

Equation (36) is the total time for the round trip of the radar pulses in terms of a measureable distance, (r) to the order of c^{-3}.

3.0 Summary and Conclusion

In this paper we used the theory of General Relativity (GR) to obtained the metric tensors in terms of the spheriodal coordinate system (η,ξ,Φ), evidence in equation (18) – (23). These metric tensors were applied to the world line element and resolved the radar sounding phenomena in the gravitational field established by the oblate sun via the world line element. The total time taken by the radar pulses to move from r_1 to r_2 and back to r_1 is given by equation (36), which is comparable with the homogenous spherical massive sun. our expression for the time delay in equation (36) is henceforth opened for experimental verification and comparison with known works.

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