ON STABILITY OF CHARTIER'S BLOCK FORMULAE

By

K.O. MUKA[1] and M.N.O. IKHILE[2]

Department of Mathematics

University of Benin

PMB 1154, Benin City, Nigeria

E-mail: kingsleymuk@yahoo.com[1] and mnoikhilo@yahoo.com[2]

Abstract

This paper focuses on obtaining stability regions of block methods developed in Chartier [8] via method of Boundary locus plot. In particular, the aim of this paper is to prove the L-stability of block methods in Chartier [8] for block size seven and eight.

Keywords: Stability region, one-block, parallel block method.

1. Introduction

Many numerical techniques are available for the solution of initial value problems (IVPs) in ordinary differential equations (ODEs) on parallel computers and these techniques depend on factors such as degree of parallelism, speed of convergence, computational expense, data-storage requirements, accuracy, and stability see, Brugnano and Trigiante [1], Burrage [2, 3], Petcu [4], Zarina et al [5]. Chu and Hamilton [6], Shampine and Watts [7] suggest that the stability problem appears to be the most serious limitation of block methods. In Chartier [8], a family of high order block methods with superior linear stability properties compared to those of Sommeijer et al [9] is developed. This family of methods is proved to be L-stable for block sizes $k \leq 6$, and conjectured to include L-stable methods of block sizes $k = 7$ and $k = 8$.

[1]**Corresponding author: Tel: +2348060482578**

2. Chartier's Block Formulae

In this section, we present the Chartier's block formulae for block size $k \le 8$. In Muka [10], k-vectors specified as

$$Y_{m-i} = \begin{pmatrix} y_{n-i\mu+1} \\ y_{n-i\mu+2} \\ \vdots \\ y_{n-i\mu+k} \end{pmatrix}, F(Y_{m-i}) = \begin{pmatrix} f(y_{n-i\mu+1}) \\ f(y_{n-i\mu+2}) \\ \vdots \\ f(y_{n-i\mu+k}) \end{pmatrix}, \quad i = 0,1; \;\; \mu = 1,2,\cdots,k-1,k,k+1,\cdots \qquad (1)$$

are introduced. If $\mu = 1$in (1), then the consecutive block overlap, this is the case in block methods developed in Sommeijer et al [9] and Chartier [8], and for $\mu = k$ in (1) is the non-overlap in consecutive block defined in Chu and Hamilton [6], and Fatunla [11]. Block definitions with $\mu = 2,3,\cdots,k-1$ and $\mu \ge k+1$have not being considered in literature. However, Muka [10] noted that solutions ${}^{[\mu]}Y_{m+1}$ generated for $\mu = 1,2,\cdots,k-1,k,k+1,\cdots$provide cheap error estimate for variable step-size implementation. That is

$$Error = \left\| {}^{[\mu+1]}Y_{m+1} - {}^{[\mu]}Y_{m+1} \right\|, \qquad \mu = 1,2,\cdots,k,k+1,\cdots \qquad (2)$$

The Chartier's block formulae is of the form

$$Y_m = AY_{m-1} + \frac{h}{\gamma}(I + diag(c))F(Y_m) \qquad (3)$$

where A is $k \times k$ coefficient matrix, I is the $k \times k$ unit matrix , diag(c) is $k \times k$ diagonal matrix with $c = (1,2,\cdots,k)^T$; h is the step-size and γ, a constant parameter chosen in such a way as to improve the order of the method (3). The matrix A in equation (3) for $k = 2(1)8$, with corresponding γ values are given as

[1]Corresponding author: Tel: +2348060482578

Two Point Block Coefficient:

$$A=\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{4} & \frac{5}{4} \end{pmatrix}, \quad \gamma = 4, \quad Y_{m-1} = \begin{pmatrix} y_n \\ y_{n+1} \end{pmatrix}, \quad Y_m = \begin{pmatrix} y_{n+1} \\ y_{n+2} \end{pmatrix}, \quad F(Y_m) = \begin{pmatrix} f_{n+1} \\ f_{n+2} \end{pmatrix}. \tag{4}$$

Three Point Block Coefficient:

$$A=\begin{pmatrix} \frac{1}{\gamma} & 1 & -\frac{1}{\gamma} \\ -\frac{3}{2\gamma} & \frac{6}{\gamma} & \frac{2\gamma-9}{2\gamma} \\ \frac{2\gamma-12}{2\gamma} & -\frac{3\gamma-16}{2\gamma} & \frac{6\gamma-20}{2\gamma} \end{pmatrix}, \quad \gamma = 3.18, \quad Y_{m-1} = \begin{pmatrix} y_n \\ y_{n+1} \\ y_{n+2} \end{pmatrix}, \quad Y_m = \begin{pmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+3} \end{pmatrix}, \quad F(Y_m) = \begin{pmatrix} f_{n+1} \\ f_{n+2} \\ f_{n+3} \end{pmatrix} \tag{5}$$

The block definitions of others are similarly obtained.

Four Point Block Coefficient:

$$A=\begin{pmatrix} \frac{2}{15} & \frac{6}{5} & -\frac{2}{5} & \frac{1}{15} \\ -\frac{1}{10} & \frac{3}{5} & \frac{7}{10} & -\frac{1}{5} \\ \frac{4}{15} & -\frac{6}{5} & \frac{12}{5} & -\frac{7}{15} \\ \frac{5}{6} & -3 & \frac{7}{2} & -\frac{1}{3} \end{pmatrix}, \quad \gamma = 5. \tag{6}$$

Five Point Block Coefficient:

$$A=\begin{pmatrix} \frac{1}{2\gamma} & -\frac{-10-6\gamma}{6\gamma} & -\frac{3}{\gamma} & \frac{1}{\gamma} & -\frac{1}{6\gamma} \\ -\frac{1}{4\gamma} & \frac{2}{\gamma} & 1 & -\frac{2}{\gamma} & \frac{1}{4\gamma} \\ \frac{1}{3\gamma} & -\frac{2}{\gamma} & \frac{6}{\gamma} & -\frac{20-6\gamma}{6\gamma} & -\frac{1}{\gamma} \\ -\frac{5}{4\gamma} & \frac{20}{3\gamma} & -\frac{15}{\gamma} & \frac{20}{\gamma} & \frac{24\gamma-250}{24\gamma} \\ \frac{24\gamma-300}{24\gamma} & -\frac{30\gamma-366}{6\gamma} & \frac{40\gamma-468}{4\gamma} & -\frac{60\gamma-642}{6\gamma} & \frac{120\gamma-924}{24\gamma} \end{pmatrix}, \quad \gamma = 4.37. \tag{7}$$

Six Point Block Coefficient:

$$A=\begin{pmatrix} \frac{2}{5\gamma} & \frac{52+24\gamma}{24\gamma} & -\frac{4}{\gamma} & \frac{2}{\gamma} & -\frac{2}{3\gamma} & \frac{1}{10\gamma} \\ -\frac{3}{20\gamma} & \frac{3}{2\gamma} & -\frac{-12-12\gamma}{12\gamma} & -\frac{3}{\gamma} & \frac{3}{4\gamma} & -\frac{1}{10\gamma} \\ \frac{2}{15\gamma} & -\frac{1}{\gamma} & \frac{4}{\gamma} & \frac{16-12\gamma}{12\gamma} & -\frac{2}{\gamma} & \frac{1}{5\gamma} \\ -\frac{1}{4\gamma} & \frac{5}{3\gamma} & -\frac{5}{\gamma} & \frac{10}{\gamma} & -\frac{130-24\gamma}{24\gamma} & -\frac{1}{\gamma} \\ \frac{6}{5\gamma} & -\frac{15}{2\gamma} & \frac{20}{\gamma} & -\frac{30}{\gamma} & \frac{30}{\gamma} & \frac{120-1644\gamma}{120\gamma} \\ -\frac{120\gamma-1918}{120\gamma} & \frac{144\gamma-2268}{24\gamma} & -\frac{180\gamma-2772}{12\gamma} & \frac{240\gamma-3556}{12\gamma} & -\frac{360\gamma-4914}{24\gamma} & \frac{720\gamma-7308}{120\gamma} \end{pmatrix}, \quad \gamma = 3.92. \tag{8}$$

[1]**Corresponding author: Tel: +2348060482578**

Seven Point Block Coefficient:

$$A=\begin{pmatrix} \frac{1}{3\gamma} & -\frac{-308-120\gamma}{120\gamma} & -\frac{5}{\gamma} & \frac{10}{3\gamma} & -\frac{5}{3\gamma} & \frac{1}{2\gamma} & -\frac{1}{15\gamma} \\ -\frac{1}{10\gamma} & \frac{6}{5\gamma} & \frac{84+48\gamma}{48\gamma} & -\frac{4}{\gamma} & \frac{3}{2\gamma} & -\frac{2}{5\gamma} & \frac{1}{20\gamma} \\ \frac{1}{15\gamma} & -\frac{3}{5\gamma} & \frac{3}{\gamma} & 1 & -\frac{3}{\gamma} & \frac{3}{5\gamma} & -\frac{1}{15\gamma} \\ -\frac{1}{12\gamma} & \frac{2}{3\gamma} & -\frac{5}{2\gamma} & \frac{20}{3\gamma} & \frac{48\gamma-140}{48\gamma} & -\frac{2}{\gamma} & \frac{1}{6\gamma} \\ \frac{1}{5\gamma} & -\frac{3}{2\gamma} & \frac{5}{\gamma} & -\frac{10}{\gamma} & \frac{15}{\gamma} & -\frac{924-120\gamma}{120\gamma} & -\frac{1}{\gamma} \\ -\frac{7}{6\gamma} & \frac{42}{5\gamma} & -\frac{105}{4\gamma} & \frac{140}{3\gamma} & -\frac{105}{2\gamma} & \frac{42}{\gamma} & \frac{720\gamma-12348}{720\gamma} \\ \frac{720\gamma-14112}{720\gamma} & -\frac{840\gamma-16304}{120\gamma} & \frac{1008\gamma-19296}{48\gamma} & -\frac{1260\gamma-23616}{36\gamma} & \frac{1680\gamma-30368}{48\gamma} & -\frac{2520\gamma-42192}{120\gamma} & \frac{5040\gamma-64224}{720\gamma} \end{pmatrix}, \quad \gamma = 5.54. \tag{9}$$

Eight Point Block Coefficient:

$$A=\begin{pmatrix} \frac{2}{7\gamma} & \frac{2088+720\gamma}{720\gamma} & -\frac{6}{\gamma} & \frac{5}{\gamma} & -\frac{10}{3\gamma} & \frac{3}{2\gamma} & -\frac{2}{5\gamma} & -\frac{1}{21\gamma} \\ -\frac{1}{14\gamma} & \frac{1}{\gamma} & -\frac{-564-240\gamma}{240\gamma} & -\frac{5}{\gamma} & \frac{5}{2\gamma} & -\frac{1}{\gamma} & \frac{1}{4\gamma} & -\frac{1}{35\gamma} \\ \frac{4}{105\gamma} & -\frac{2}{5\gamma} & \frac{12}{5\gamma} & \frac{144+144\gamma}{144\gamma} & -\frac{4}{\gamma} & \frac{6}{5\gamma} & -\frac{4}{15\gamma} & \frac{1}{35\gamma} \\ -\frac{1}{28\gamma} & \frac{1}{3\gamma} & -\frac{3}{2\gamma} & \frac{5}{\gamma} & -\frac{180-144\gamma}{144\gamma} & -\frac{3}{\gamma} & \frac{1}{2\gamma} & -\frac{1}{21\gamma} \\ \frac{2}{35\gamma} & -\frac{1}{2\gamma} & \frac{2}{\gamma} & -\frac{5}{\gamma} & \frac{10}{\gamma} & \frac{240\gamma-1128}{240\gamma} & -\frac{2}{\gamma} & \frac{1}{7\gamma} \\ -\frac{1}{6\gamma} & \frac{7}{5\gamma} & -\frac{21}{4\gamma} & \frac{35}{3\gamma} & \frac{35}{2\gamma} & \frac{21}{\gamma} & -\frac{7308-720\gamma}{720\gamma} & -\frac{1}{\gamma} \\ \frac{8}{7\gamma} & -\frac{28}{3\gamma} & \frac{168}{5\gamma} & -\frac{70}{\gamma} & \frac{280}{3\gamma} & -\frac{84}{\gamma} & \frac{56}{\gamma} & \frac{5040\gamma-104544}{5040\gamma} \\ -\frac{5040\gamma-117612}{5040\gamma} & -\frac{5760\gamma-133488}{720\gamma} & -\frac{6720\gamma-154296}{240\gamma} & \frac{8064\gamma-182736}{144\gamma} & -\frac{10080\gamma-223884}{144\gamma} & \frac{13440\gamma-288432}{240\gamma} & -\frac{20160\gamma-402408}{720\gamma} & \frac{40320\gamma-623376}{5040\gamma} \end{pmatrix}, \gamma = 7.25. \tag{10}$$

3. **STABILITY ANALYSIS**

The linear stability of (3) can be investigated by applying it to the standard linear test problem

$$y' = \lambda y, \qquad \lambda < 0 \tag{11}$$

This will give rise to the recurrence equation

$$Y_m = M(z)Y_{m-1}, \quad z = \lambda h, \tag{12}$$

where

[1]**Corresponding author: Tel: +2348060482578**

$$M(z) = (I - \frac{z}{\gamma}(I + diag(c))^{-1} A \qquad (13)$$

is the amplification matrix.

Definition 3.1

The region of absolute stability R_A *of block method (3) is* $\mathrm{R}_A = \{z \in \mathbb{C} : |\rho(M(z))| \le 1\}$*,* *where* $\rho(M(z))$ *is the spectra radius of the amplification matrix* $M(z)$ *and roots corresponding to* $|\rho(M(z))| = 1$ *is simple.*

Definition 3.2

A k-dimensional block method is said to be zero stable if the region of absolute stability $R_A \supseteq \{z \in \mathbb{C} : |\rho(M(0))| \le 1\}$*.*

Definition 3.3

A k-dimensional block method is A-stable if the region of absolute stability $R_A \supseteq \mathbb{C}^-$.

Definition 3.4 (cf. Chartier [8])

A k-dimensional block method is L-stable if it is A–stable and $\rho(M(\infty)) = 0$.

An interesting feature in Chartier block methods is the fact that the eigenvalues of matrix A are known a priori and are given as

$$\lambda(\gamma) = 1 - \frac{j}{\gamma}, \qquad j = 0,1,\cdots k-1, \quad \gamma > 0. \qquad (14)$$

The (3) for k=2(1)8 is zero-stable. If the method (3) is A-stable, it automatically implies that it is L-stable since

$$\lim_{|z| \to \infty} M(z) = \lim_{|z| \to \infty} (I - \frac{z}{\gamma}(I + diag(c))^{-1} A = 0 \qquad (15)$$

[1]**Corresponding author: Tel: +2348060482578**

holds. The first and most direct means in finding the region of absolute stability is the root locus plot method see, Lambert [12], Muka and Ikhile [13,14], another means and widely used in literature in determining region of absolute stability of numerical techniques is the method of boundary locus Fatunla [15], Lambert [16], Hairer et al [17]. The absolute stability region R_A associated with the Chartier's formulae (3) with coefficients (4)–(10) are presented in the boundary locus plots below:

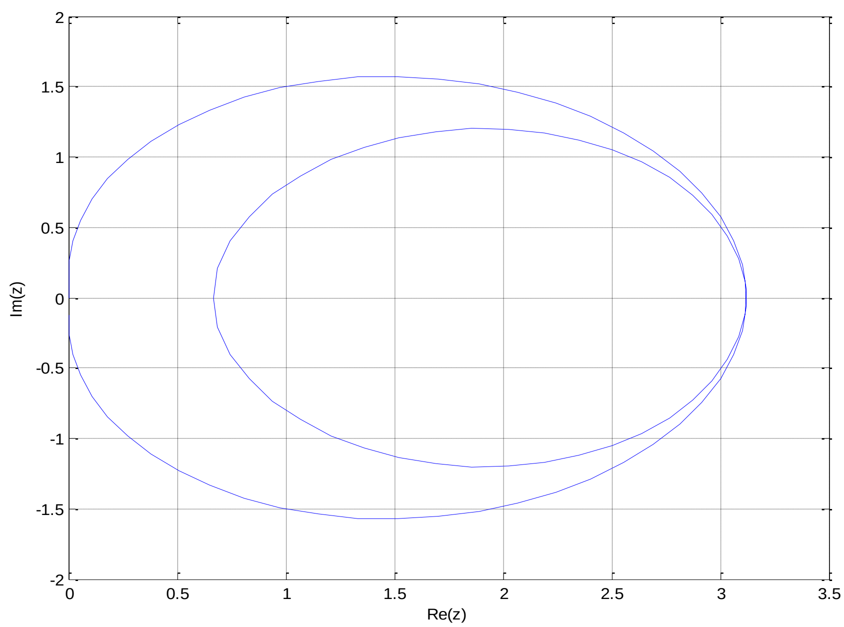

Fig. 1: Stability Plot of Chartier's formulae with Coefficient (4)

[1]**Corresponding author: Tel: +2348060482578**

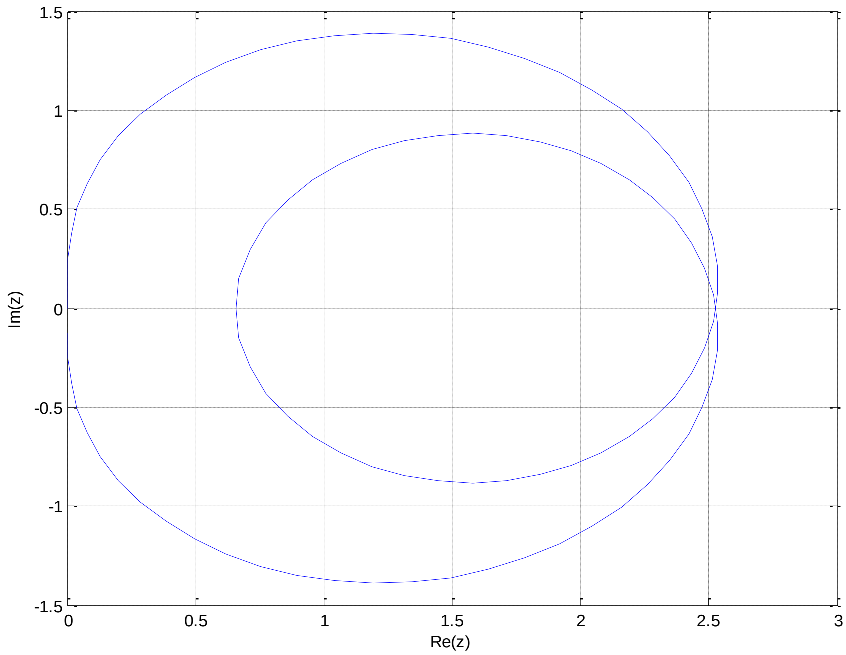

Fig. 2: Stability Plot of Chartier's formulae with Coefficient (5)

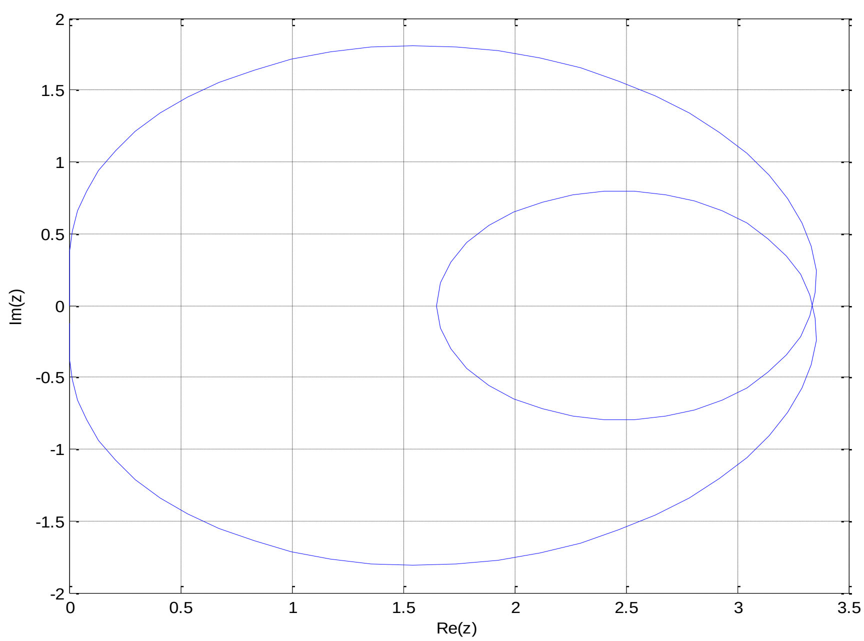

Fig. 3: Stability Plot of Chartier's formulae with Coefficient (6)

[1]**Corresponding author: Tel: +2348060482578**

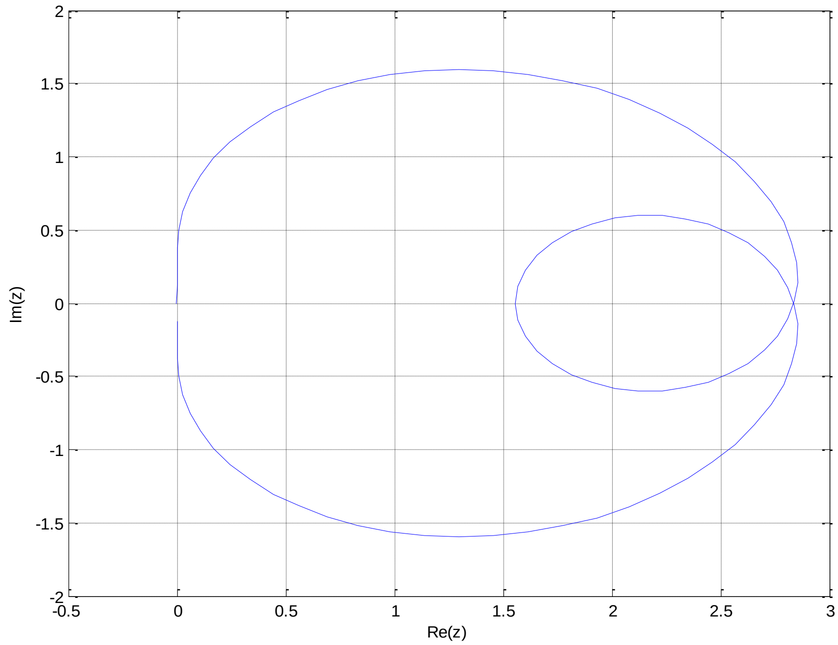

Fig. 4: Stability Plot of Chartier's formulae with Coefficient (7)

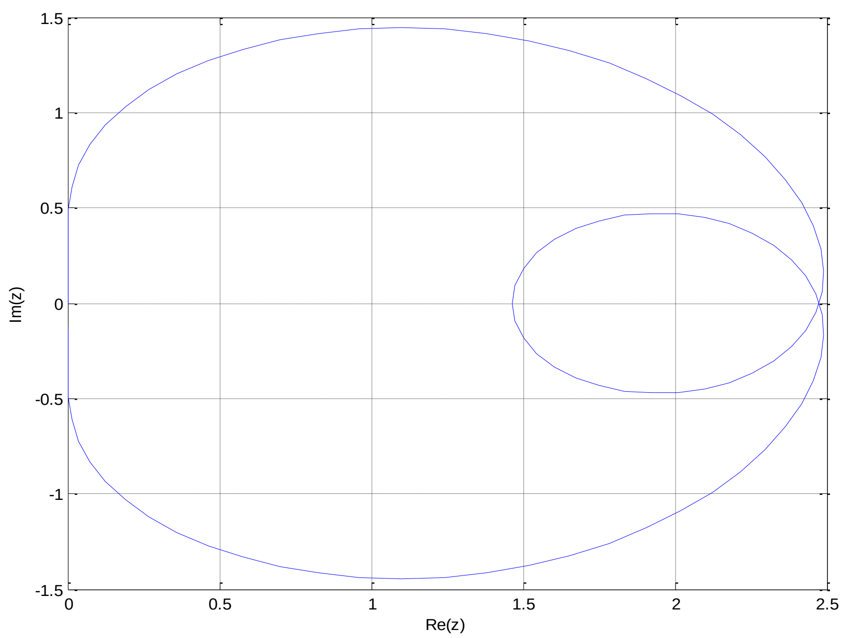

Fig. 5: Stability Plot of Chartier's formulae with Coefficient (8)

[1]Corresponding author: Tel: +2348060482578

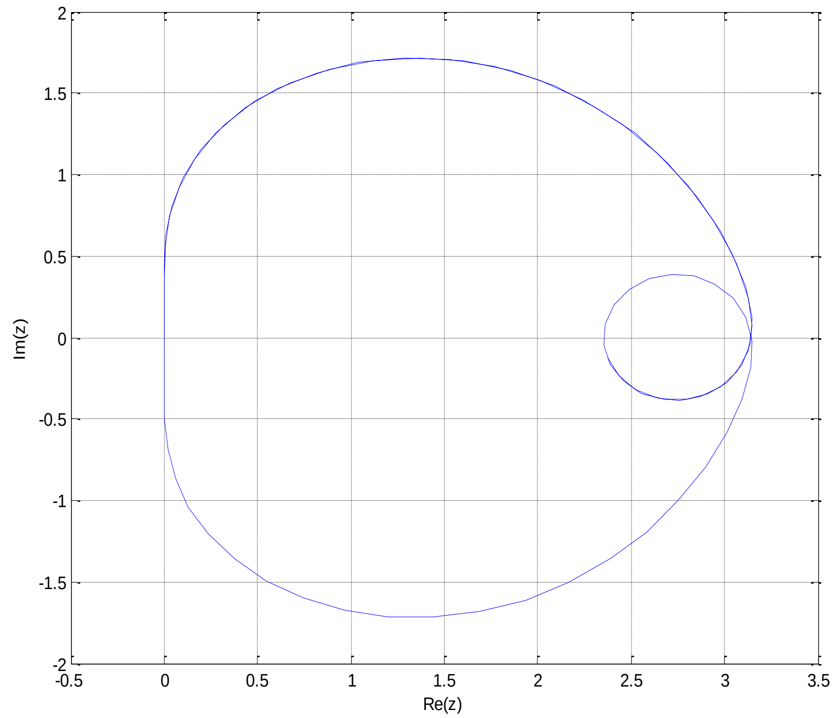

Fig. 6: Stability Plot of Chartier's formulae with Coefficient (9)

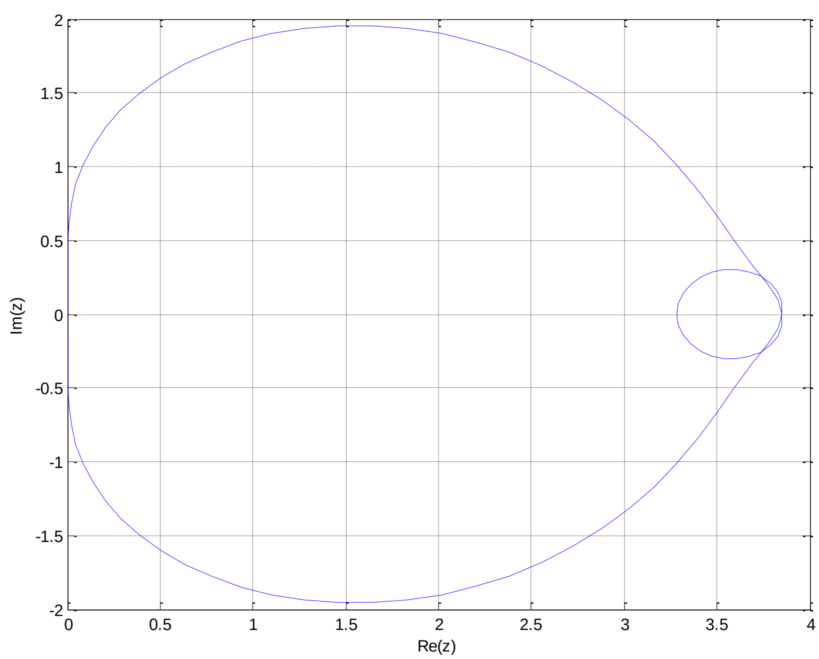

Fig. 7: Stability Plot of Chartier's formulae with Coefficient (10)

[1]Corresponding author: Tel: +2348060482578

Stable regions are points outside the enclosed curves. Since in figures (1)–(7), the entire left of the complex plane for block-sizes $k \leq 8$ are contained in the stable region, it therefore follows that Chartier's block formulae (3) is L–stable for block-sizes $k \leq 8$. In Chartier [8], L-stability of (3) for k=7 and k=8 are guessed to be L-stable but with the boundary locus plots in figures 6 and figure 7 this guess is proven to be true.

4. **Numerical Experiment**

The suitability of Chartier's block formulae for the integration of stiff IVPs in ODEs can be found in Chartier [8]. To further illustrate this, we integrate the test problem below using two-point Chartier's block formulae (3) with step-size h=0.2

Problem (cf. Zarina et al [5])

$$y' = f(t, y) = -10y; \qquad y(2) = e^{-20}; \quad -2 \leq t \leq 3$$

$$Exact\ Solution \quad y(t) = e^{-10t}$$

Maximum Error $= 2.10278 \times 10^{-11}$.

If the stiff problem is solved using explicit Euler method, the restriction on the step-size h is $|\lambda h| < 2.$ Therefore the problem cannot be solved with explicit Euler for h=0.2. The problem solved with Chartier's block formulae for h=0.2, show that they are suitable for stiff problems because of their L-stability properties.

5. Conclusion

Chartier's conjecture is proved herein using the boundary locus plot method and simple numerical test performed to further illustrate their suitability for the integration of Stiff IVPs in ODEs.

[1]**Corresponding author: Tel: +2348060482578**

References

[1] Brugnano, L. and Trigiante, D. (2001), Block implicit methods for ODEs, in recent trends in numerical analysis, Trigiante. D ed., pp. 81-105. Nova science pub. Inc., New York.

[2] Burrage, K. (1993), Parallel solution of ODEs by multiblock methods, SIAM J. Sci. Stat. Comp., Vol. 8 , pp.342 -535.

[3] Burrage, K. (1995), Parallel and sequential methods for ordinary differential equations, Oxford University press Inc., New York.

[4] Petcu, D. (1998), Parallelism in solving ordinary differential equations, Mathematical monographs 64, Tipografia universitatii.

[5] Zarina, B. I., Rozita, J. and Fudzaiah, I. (2003), On stability of fully implicit block backward differentiation formula. Matematika Vol 19(2) ; pp.83 - 89

[6] Chu, M. T. and Hamilton, H. (1987), Parallel solution of ODEs by multiblock methods, SIAM J. Sci. Stat. Comp., Vol. 8 , pp.342 -535.

[7] Shampine, L. F. and Watts, H. A. (1969), Block implicit one step methods. Math. Of comp. Vol. 23: pp. 731-740.

[8] Chartier, P. (1993), Parallelisme dans la resolution numerique des problems de valeur initial pour les equations differentielles ordinaries et algebriques. Ph.D Thesis, univ. Of Rennes, France.

[9] Sommeijer, B. P., Couzy, W. and Houwen, P.J. (1989), A – Stable parallel block methods, Report NM – R8918, Center for Math. And Comp. Sci., Amsterdam.

[10] Muka, K. O. (2011), Second derivative parallel block methods for initial value problems in ordinary differential equations. Ph.D Thesis, Univ. Of Benin, Nigeria.

[1]Corresponding author: Tel: +2348060482578

[11] Fatunla, S. O. (1990), Block methods for second order ODEs. Inter. J. Comp. maths. Vol. 14; pp. 55–63.

[12] Lambert, J.D. (1973), Computational methods in ordinary differential equations, John Wiley & Sons, London, UK.

[13] Muka, K. O. and Ikhile, M.N.O. (2009), Second derivative parallel block backward differentiation type formulas for stiff Odes. Journ. Nig. Assc. Of Maths. Physics. Vol. 14, pp. 117 -124.

[14] Muka, K. O. and Ikhile, M.N.O. (2009), Generalized Enright block methods for stiff Odes. Journ. Nig. Assc. Of Maths. Physics. Vol. 14, pp. 125 -134.

[15] Fatunla, S. O. (1988), Numerical methods for initial value problems in ordinary differential equation. Academic press, inc. UK.

[16] Lambert, J.D. (1991), Numerical methods for ordinary differential system: the initial value problems, John Wiley & Sons, Chichester.

[17] Hairer, E., Norsett, S. and Wanner, G. (2002), Solving ordinary differential equations II. Siff and Differential – Algebraic problems. Vol. 2, Springer – verlag.

[1]**Corresponding author: Tel: +2348060482578**