AN INVENTORY MODEL FOR DETERIORATING ITEMS WITH TIME DECREASING DEMAND AND PARTIAL BACKLOGGING

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ABSTRACT

This paper proposes an Economic Order quantity (EOQ) inventory model for deteriorating items with time dependent demand and constant deterioration. In this model, shortage is allowed and partially backlogged. The backlogging rate is variable and dependent on the waiting time for the next replenishment. Mathematica 7.5 was used to simplify the mathematica complexity of the various differential equation. Numerical example is used to illustrate the proposed model. Sensitivity analysis is also carried out.

Keywords: Inventory; deteriorating item; partial backlogging; shortages.

1. INTRODUCTION

The deterioration of goods is a realistic phenomenon in many inventory systems. Therefore, controlling and regulating deterioration of items is a major problem in any inventory system. In general, deterioration is defined as damage, spoilage, decay, obsolescence, evaporation, pilferage etc. that result in decrease of usefulness of the original one. It is plausible to note that a product may be understood to have a lifetime which ends when utility reaches zero. For items such as steel, hardware, glass ware and toys, the rate of deterioration is so low that there is little need for considering deterioration in the determination of the economic lot size. However, some items such as blood, fish, strawberry, alcohol, gasoline, radioactive chemical, medicine and food items deteriorate remarkably overtime.

Whitin [1] considered an inventory model for fashion goods deteriorating at the end of a prescribed storage period. Ghare and Scharder [2] developed an EOQ models with an exponential decay and a deterministic demand. Covert and Philip [3] proposed EOQ models for deterioration which follows Weibull distribution. Mandal and Phaujdar [4] developed a model for deteriorating items, constant and varying with time and a stock dependent consumption rate. Also Dave and Patel [5], Sachan [6] developed EOQ models for linear trend in demand. Mak [7], Aggrawal and Bahari-Kashani [8] develope EOQ models to allow deterioration and an exponential trend in demand.

Further, Ouyang [9] proposed an EOQ model for items with constant deterioration. Demand was assumed to be both constant and exponentially decreasing and shortage partially backlogged. Basu and Sinha [10] developed a model for time dependent deterioration and partial backlogging of shortages. Arya [11] proposed an optimisation framework to derive optimal replenishment policy for deteriorating items with stock dependent demand. Mirzazadeh [12] developed a model for the optimal production of an inventory control system of deteriorating items under time-varying and stochastic inflation environment. Unsatisfied demand is partially backlogged and the cost component of the inventory system is considered using their expected present values.

In this paper, we develop a model by taking deterioration as constant with time, demand exponentially decreasing and define the rate of partial backlogging as dependent on the waiting time for the next replenishment. This paper is a modification to Ouyang et al [9] by assuming that the partial backlogging rate is defined as $e^{-\delta(T-t)}$; where δ is a positive constant.

2. NOTATIONS

C_1: Holding cost per unit

C_2: Deterioration cost per unit

C_3: Shortage cost per unit

C_4: Opportunity cost due to lost sales per unit

C_5: Ordering cost per unit

t_1 : Time at which shortage starts

T: Length of each ordering cycle

W: The maximum inventory for each ordering cycle

S: The maximum amount of demand backlogged for each ordering cycle

Q: The order quantity for each ordering cycle

I(t): The inventory level at time t.

3. Assumptions

1) The inventory system involves only one item and the planning horizon is infinite.

2) The replenishment occurs instantaneously at an infinite rate.

3) The rate of deterioration is constant i.e θ and there is no replacement of deteriorated units during the period under consideration.

4) The demand rate $D(t) = \alpha e^{-\beta t}$ is known, where $\alpha > 0$ is the initial demand and β is a constant governing the decreasing rate of demand.

5) During the shortage period, the backlogging rate is variable and is dependent on the length of waiting time for the next replenishment. The longer the waiting time the smaller the backlogging rate would be. Hence, the proportion of customers who would like to accept backlogging at time t is decreasing with waiting time (T-t) waiting for the next replenishment.

We have thus defined the backlogging rate to be $e^{-\delta(T-t)}$ when inventory is negative. The backlogging parameter δ is a positive constant defined within the interval $t_1 \le t \le T$

3. THE MATHEMATICAL MODEL

The behavior of the inventory system at any time is depicted in fig. 1

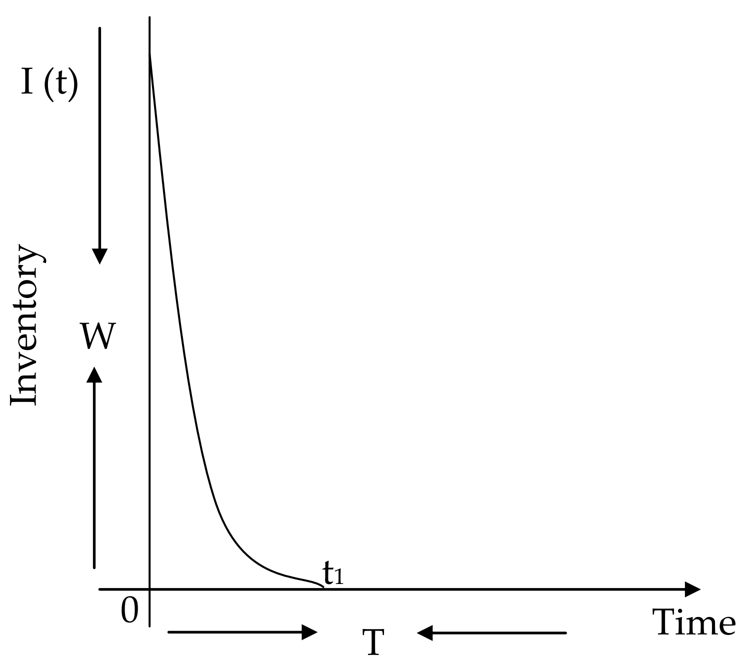

Figure 1: Inventory vs Time

Notice that replenishment for the above inventory system is made at time t = 0 and the inventory level is at its maximum, W. The inventory level decreases due to both market demand and deterioration during the period $[0, t_1]$ and is zero at $t = t_1$. Shortage is allowed to occur during the time interval $[t_1, T]$ and all demand during the period is partially backlogged.

As described above, the inventory system declines owing to demand and deterioration in $[0, t_1]$. Hence, the differential equation representing the inventory status is given by:

$$\frac{dI_1(t)}{dt} + \theta I_1(t) = -\alpha \ell^{-\beta t}, \ 0 \le t \le t_1 \qquad (1.0)$$

With boundary condition that $I_1(0) = W$. The solution to equation (1.0) is given as:

$$I_1(t) = \frac{\alpha}{\theta - \beta} \ell^{(\theta-\beta)t_1 - \theta t} - \frac{\alpha}{\theta - \beta} \ell^{-\beta t}$$

$$I_1(t) = \frac{\alpha}{\theta - \beta} \left[\ell^{(\theta-\beta)t_1 - \theta t} - \ell^{-\beta t} \right] \qquad (1.1)$$

The maximum inventory level for each ordering cycle is given as $I_1(0) = W$. Thus,

$$W = \frac{\alpha}{\theta - \beta} \left[e^{(\theta-\beta)t_1} - 1 \right] \qquad (1.2)$$

During the shortage interval $[t_1, T]$, the demand at any time t is partially backlogged at $e^{-\delta(T-t)}$, $t_1 \le t \le T$. Thus, the differential equation governing the amount of demand backlogged is:

$$\frac{dI_2(t)}{dt} = -\alpha e^{-\beta t} . e^{-\delta(T-1)}, \qquad t_1 \le t \le T \qquad (1.3)$$

with the boundary condition that $I_2(t_1) = 0$. The solution to equation (1.3) is given as

$$I_2(t) = -\frac{\alpha}{\delta - \beta} e^{(\delta-\beta)t - \delta T} + C, \qquad (1.4)$$

Given that $I_2(t_1) = 0$, we have that

$$C = \frac{-\alpha}{\delta - \beta} e^{(\delta-\beta)t_1 - \delta T} \qquad (1.5)$$

Substituting (1.5) into (1.4) we have that

$$\frac{\alpha}{\delta-\beta}\left[e^{(\delta-\beta)t_1-\delta T}-e^{(\delta-\beta)t-\delta T}\right] \tag{1.6}$$

If we let t = T in (1.6) we obtain the maximum amount of demand backlogged per cycle. This is given as S , where

$$S=\frac{\alpha}{\delta-\beta}e^{(\delta-\beta)t_1-\delta T}-\frac{\alpha}{\delta-\beta}e^{\beta T} \tag{1.7}$$

If $M=\frac{\alpha}{\delta-\beta}$

then equation (1.7) becomes

$$S=M\left[\ell^{(\delta-\beta)t_1-\delta T}-\ell^{-\beta T}\right] \tag{1.8}$$

Hence the order quantity per cycle Q is

$$Q=W+S$$

For convenience we let $X=\frac{\alpha}{\theta-\beta}$

Hence,

$$Q=X\left[\ell^{(\theta-\beta)t_1}-1\right]+M\left[\ell^{(\delta-\beta)t_1-\delta T}-\ell^{-\beta T}\right] \tag{1.9}$$

The various cost components are calculated as follows:

We define our holding cost HC as

$$HC=\int_o^{t_1} C_1 I_1(t)dt$$

$$HC=C_1\int_o^{t_1} X\ell^{(\theta-\beta)t_1-\theta t}dt-C_1\int_o^{t_1} X\ell^{-\beta t}dt$$

$$HC=\frac{XC_1}{\theta}\left[e^{(\theta-\beta)t_1}-e^{\beta t_1}\right]+\frac{XC_1}{\beta}\left[e^{-\beta t_1}-1\right] \tag{1.10}$$

We define the deterioration cost DC as

$$DC=C_2\left[W-\int_0^{t_1} D(t)dt\right]$$

$$DC = C_2\left[W - \int_0^{t_1} \alpha \ell^{-\beta t} dt\right]$$

$$= C_2\left[X\left(e^{(\theta-\beta)t_1} - 1\right) + \frac{\alpha}{\beta}\left(e^{-\beta t_1} - 1\right)\right] \quad (1.11)$$

The Shortage cost SC for the inventory system is defined as

$$SC = C_3\left[-\int_{t_1}^{T} I_2(t)dt\right]$$

$$SC = C_3\left[-\int_{t_1}^{T}\left(M\ell^{(\delta-\beta)t_1-\delta T} - M\ell^{(\delta-\beta)t-\delta T}\right)dt\right]$$

$$SC = C_3\left[Me^{(\delta-\beta)t_1-\delta T}\left(T - t_1\right) + \frac{M}{\delta-\beta}e^{-\delta T}\left(e^{(\delta-\beta)t_1} - e^{(\delta-\beta)T}\right)\right] \quad (1.12)$$

Opportunity cost due to lost sales BC

$$BC = C_4\int_{t_1}^{T}\left(1 - \ell^{-\delta(T-t)}\right)\alpha\ell^{-\beta t} dt$$

$$BC = C_4\left[\frac{\alpha}{\beta}\left(e^{-\beta t_1} - e^{\beta T}\right)\right] - MC_4\left[e^{-\beta T} - e^{-\delta T+(\delta-\beta)t_1}\right] \quad (1.13)$$

Therefore the total cost per unit time per cycle TC is given as:

$$TC = \frac{1}{T}\left[\begin{array}{l} \frac{XC_1}{\theta}\left[e^{(\theta-\beta)t_1} - e^{-\beta t_1}\right] + \frac{XC_1}{\beta}\left[e^{-\beta t_1} - 1\right] + C_2\left[X\left(e^{(\theta-\beta)t_1} - 1\right) + \frac{\alpha}{\beta}\left(e^{-\beta t_1} - 1\right)\right] + \\ C_3\left[M..e^{(\delta-\beta)t_1-\delta T}\left(T - t_1\right) + \frac{M}{\delta-\beta}e^{-\delta T}\left(e^{(\delta-\beta)t_1} - e^{(\delta-\beta)T}\right)\right] + C_4\left[\frac{\alpha}{\beta}\left(e^{-\beta t_1} - e^{-\beta T}\right)\right] \\ -MC_4 + C_5\left[e^{-\beta t_1} - e^{-\delta T+(\delta-\beta)t_1}\right] + C_5 \end{array}\right] \quad (1.14)$$

The objective of the model is to determine the optimal values of t_1^* and T^* in order to minimize the average total cost per unit time. The optimal solutions t_1^* and T^* need to satisfy the following equations.

$$\frac{\delta TC}{\delta t_1} = \frac{1}{T}(-e^{-\beta t_1}\alpha C_4 + e^{-T\delta+(-\beta+\delta)t_1}\alpha C_4 + e^{-T\delta+(-\beta+\delta)t_1}(-\beta+\delta)C_3(T-t_1)W_1 - e^{-\beta t_1}C_1X_1 + \frac{(e^{-\beta t_1}\beta + e^{(-\beta+\theta)t_1}(-\beta+\theta))C_1X_1}{\theta} + C_2(-e^{-\beta t_1}\alpha + e^{(-\beta+\theta)t_1}(-\beta+\theta)X_1)) \quad (1.15)$$

Note that M = W_1 and X = X_1 in equations (1.15) and (1.16)

$$\frac{\delta TC}{\delta T} = \frac{1}{T}(e^{-T\beta}\alpha C_4 - \frac{\alpha(-e^{-T\beta}\beta + e^{-T\delta+(-\beta+\delta)t_1}\delta)C_4}{-\beta+\delta} + C_3(-e^{-T\delta+T(-\beta+\delta)}W_1 + e^{-T\delta+(-\beta+\delta)t_1}W_1 - \frac{e^{-T\delta}(-e^{T(-\beta+\delta)}+e^{(-\beta+\delta)t_1})\delta W_1}{-\beta+\delta} - e^{-T\delta+(-\beta+\delta)t_1}\delta(T-t_1)W_1)) - \frac{1}{T^2}((-\frac{e^{-T\beta}\alpha}{\beta} + \frac{e^{-\beta t_1}\alpha}{\beta})C_4 - \frac{(e^{-T\beta}-e^{-T\delta+(-\beta+\delta)t_1})\alpha C_4}{-\beta+\delta} + C_5 + C_3(\frac{e^{-T\delta}(-e^{T(-\beta+\delta)}+e^{(-\beta+\delta)t_1})W_1}{-\beta+\delta} + e^{-T\delta+(-\beta+\delta)t_1}(T-t_1)W_1) + \frac{(-1+e^{-\beta t_1})C_1X_1}{\beta} + \frac{(-e^{-\beta t_1}+e^{(-\beta+\theta)t_1})C_1X_1}{\theta} + C_2(\frac{(-1+e^{-\beta t_1})\alpha}{\beta} + (-1+e^{(-\beta+\theta)t_1})X_1)) \qquad (1.16)$$

Provided that they satisfy the sufficient conditions

$$\frac{\delta^2 TC}{\delta t_1^2} > 0,\ \frac{\delta^2 TC}{\delta T^2} > 0 and\ \frac{\delta^2 TC}{\delta T^2}\frac{\delta^2 TC}{\delta t_1^2} - \left(\frac{\delta^2 TC}{\delta t_1 \delta T}\right) > 0$$

4.0 Numerical examples and sensitivity analysis

In this section, we give example to illustrate our proposed model

$\alpha = 12,\ \delta = 2, C_3 = 2.5,\ \beta = 0.03,\ C_1 = 0.5,\ C_4 = 2, \theta = 0.08,\ C_2 = 1.5,\ C_5 = 10$

The result, from proposed model are

$t_1^* = 1.38164,\ T^* = 1.95245,\ Q^* = 21.04461, W^* = 17.1658,\ S^* = 3.87881,\ TC = 10.3373$

These results are obtained by the use of mathematica 7.5.

Sensitivity analysis

Table 1: Sensitivity of t_1^*, T^*, Q^*, W^* and TC^* to changes in the partial backlogging and deterioration parameters

Parameter	t_1^*	T^*	Q^*	W^*	TC^*
$\delta=2.5$	1.44472	1.87417	21.9809	17.9782	10.8371
$\delta=2$	1.38164	1.95245	21.04461	17.1658	10.3373
$\delta=1.5$	1.25847	2.08294	20.97337	15.5869	9.37787
$\delta=1$	0.0638181	0.587141	10.70808	0.76704	16.8224
$\delta=0$	1.87276	1.37172	29.2858	23.5589	14.3173
$\theta=0.08$	1.38164	1.95245	21.04461	17.1658	10.3373
$\theta=0.04$	1.50654	2.05157	21.98567	18.2153	9.89151
$\theta=0.02$	1.57873	2.11016	22.50669	18.796	9.65096
$\theta=0.5$	0.75474	1.51395	15.38771	10.8713	13.2857
$\theta=0.8$	0.573727	1.41548	13.3826	8.65648	14.4044

5. Remarks

On the basis of the sensitivity analysis we make the following observations:

By varying the value of the deteriorating parameter θ we seek to investigate the behavior of our decision variable in the proposed model. Clearly, with an increase in θ we see that there is a reduction in Q^*, W^*, t_1^* and T^* which follow aprori expectation. Also we observe that the total inventory cost TC^* increases with an increase in the rate of deterioration .In other words, the cost of maintaining inventory increase with an increase in the deterioration rate of inventory items.

By varying the partial backlogging parameter δ we observe that T^* and Q^* decreases, while t_1^*, W^* and TC^* increase with an increase in δ. Note that if $\delta=0$ we have the case of no backlogging and $\delta=1$ is indicative of complete backlog of demand during the shortage period.

6. **Conclusion**

In this paper, we propose an EOQ model for items that follow constant deterioration and is exponentially decreasing with time. We defined the rate of partial backlogging to be an exponentially decreasing function of the waiting time, which is more realistic. The optimal inventory cost of the proposed model shows a decrease when compared to Ouyang et al [9] which is a reflection of the pattern of partial backlogging. The proposed more also shows that an increase in the rate of partial backlogging will push up inventory cost. Hence, an understanding and optimal specification of this rate is important for the efficient management of the inventory system.

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