# 3 steps continuous block predictor corrector method for the solution of general second order ordinary differential equation

**Adesanya, A. Olaide, Odekunle, M. Remilekun**
Department of Mathematics, Modibbo Adama University of Technology, Yola, Adamawa State, Nigeria
+2348086049694, torlar10@yahoo.com
**Anake, T. Ashibel**
Department of Mathematics, Covenant University, Sango Ota, Ogun State, Nigeria

**Abstract**

We propose three steps non self starting block method based on collocation of the differential system and interpolation of power series approximate solution to derive a continuous linear multistep method. Continuous block formula was adopted to give the independent solution within the interval of integration. Our method was found to give better approximation than the self starting block method when tested on numerical examples.

**Key words:** collocation, interpolation, approximate solution, differential system, continuous block formula, discrete block formula, basic properties

**AMS Subject classification:** 65L05, 65L06, 65D30

## 1.0 Introduction

This paper considers the method of solving general second order ordinary differential equation of the form

$$y'' = f(x, y, y').y(x_0) = y_0, y'(x_0) = y'_0 \tag{1}$$

Where $f$ is continuous within the interval of integration.

Conventionally, (1) is solved by method of reduction to a system of first order ordinary differential equations. The disadvantages of the method of reduction is extensively discussed by Awoyemi and Kayode[1], Jator[2], Adesanya et al. [3].

Implicit Linear multistep method is applied in predictor-corrector mode where separate predictors are required to implement the corrector. The major setback of this method is that the predictors are reducing order predictors; this affects the accuracy of the method. Other setbacks of predictor corrector method are discussed by Awoyemi et al. [4], Adesanya et al.

[5]. In order to cater for the setback of predictor-corrector method, scholars such as Jator[2], Jator and Li[6], Abbas[7], Awoyemi et al.[4], Fudziah et al.[8], Fatunla[9] to mention few, proposed a discrete self starting block method of the form

$$A^{(0)}Y_m = ey_n + h^{\mu} df\left(y_n\right) + h^{\mu} bf\left(Y_m\right) \tag{2}$$

With the prediction equation of the form

$$A^{(0)}Y_m = ey_n + h^{\mu} df\left(y_n\right) \tag{3}$$

Substituting (3) into (2) gives

$$A^{(0)}Y_m = ey_n + h^{\mu} df\left(y_n\right) + h^{\mu} bf\left(Y^{(0)}{}_m\right) \tag{4}$$

Equation (4) is called the self starting block formula because the prediction equation is gotten directly from the block formula.

In this paper, we proposed a continuous block formula which enables us to make evaluation at all points within the interval of integration. The continuous block method has the same advantage as the continuous linear multistep method which is extensively discussed by Awoyemi[10], Adesanya et al.[5], Jator[2]. The non self starting property of this method makes it better in term of accuracy than the self starting method.

## 2.0 Methodology

We consider a power series approximate solution of the form

$$y(x) = \sum_{j=0}^{r+s-1} a_j x^j \tag{5}$$

Where $r$ and $s$ are the number of interpolation and collocation points respectively. The second derivative of (5) gives

$$y''(x) = \sum_{j=2}^{r+s-1} j(j-1) a_j x^{j-2} \tag{6}$$

Putting (6) in (1) gives

$$\sum_{j=2}^{r+s-1} j(j-1)a_j x^{j-2} = f(x,y,y') \tag{7}$$

Collocating (7) and interpolating (5) at selected points gives a system of non linear equation in the form

$$AU = X \tag{8}$$

Where $U$ is a $n \times n$ matrix, $A = [a_0, a_1, ..., a_{r+s-1}]^T$ and $[y_n, y_{n+1}, ..., y_{n+r}, f_n, f_{n+1}, ..., f_{n+s}]^T$

Solving for $a_j$'s in (8) and substituting back into (5) gives a continuous linear multistep method in the form

$$y(x) = \sum_{j=0}^{r} \alpha_j(x) y_{n+j} + h^{\mu} \sum_{j=0}^{s} \beta_j(x) f_{n+j} \tag{9}$$

Where $\mu$ is the order of the differential equation, $y_{n+j} = y(x_n + jh)$, $f_{n+j} = f((x_n + jh), y(x_n + jh), y'(x_n + jh))$, $\alpha_j(x)$ and $\beta_j(x)$ are polynomial and $h$ is the step-size.

Equation (9) can be written in normalized form as

$$Y_k = BY_{k-1} + h^{\mu} Cf(y_n) + h^{\mu} Gf(y_k) \tag{10}$$

Where $B$ is $s \times s$ matrix, $C$ is $r \times 1$ vector matrix, $G$ is $r \times r$ matrix and $Y_{k-1} = [y_n, y_{n+1}, ..., y_{n+k-1}]^T$

Solving (10) for the independent solution gives a continuous block method formula in the form

$$y_{n+j} = \sum_{j=1}^{\mu-1} \frac{(jh)^m}{m!} y_n^{(m)} + h^{\mu} \sum_{j=0}^{r} \sigma_j(x) f_{n+j} \tag{11}$$

Where $m = 1,2,3,..., \mu - 1$

Evaluating (11) at selected grid points reduces to the form (4). We then propose a prediction equation of the form

$$Y_m^{(0)} = ey_n + \sum_{\lambda=0}^{m} h^{\mu+\lambda} f^{\lambda}(y_0) \tag{12}$$

Where $f^{\lambda}(y_0) = \frac{\partial^{\lambda}}{\partial x^{\lambda}} f(x,y,y')_{(x_0,y_0,y'_0)}$. In this paper, we take $\lambda = 2$.

Substituting (12) into (4) gives

$$Y_m = ey_n + h^{\mu} Df(y_n) + h^{\mu} Bf(Y_0^m) \tag{13}$$

Equation (13) is our non-self starting block method since the first and second derivatives of the function are not gotten directly from the block formula as claimed by Abbas[7].

Interpolating (2) at $x_{n+r}, r = 1,2$ and collocating (7) at $x_{n+s}, s = 0(1)3$ gives a system of equation of the form (8), where

$$u = \begin{bmatrix} 1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & x_{n+1}^4 & x_{n+1}^5 \\ 1 & x_{n+2} & x_{n+2}^2 & x_{n+2}^3 & x_{n+2}^4 & x_{n+2}^5 \\ 0 & 0 & 2 & 6x_n & 12x_n^2 & 20x_n^3 \\ 0 & 0 & 2 & 6x_{n+1} & 12x_{n+1}^2 & 20x_{n+1}^3 \\ 0 & 0 & 2 & 6x_{n+2} & 12x_{n+2}^2 & 20x_{n+2}^3 \\ 0 & 0 & 2 & 6x_{n+3} & 12x_{n+3}^2 & 20x_{n+3}^3 \end{bmatrix},\ A = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix},\ B = \begin{bmatrix} y_{n+1} \\ y_{n+2} \\ f_n \\ f_{n+1} \\ f_{n+2} \\ f_{n+3} \end{bmatrix} \tag{14}$$

Solving (14) for $a_j's, j = 0(1)5$ using Gaussian elimination method and substituting back into the approximate solution, (9) reduces to

$$y(x) = \sum_{j=0}^{2} \alpha_j(x) y_{n+j} + h^2 \sum_{j=0}^{3} \beta_j(x) f_{n+j} \tag{15}$$

Where the coefficients of $y_{n+j}$ and $f_{n+j}$ gives

$$\alpha_1 = 2 - t,\ \alpha_2 = t - 1$$

$$\beta_0 = \frac{-1}{360}\left(3t^5 - 30t^4 + 110t^3 - 180t^2 + 127t - 30\right)$$

$$\beta_1 = \frac{1}{120}\left(3t^5 - 25t^4 + 60t^3 - 138t + 100\right)$$

$$\beta_2 = \frac{-1}{120}\left(3t^5 - 20t^4 + 30t^3 - 3t - 10\right)$$

$$\beta_3 = \frac{1}{360}\left(3t^5 - 15t^4 + 20t^3 - 8t\right)$$

Where $t = \frac{x - x_n}{h}$. Solving for the independent solution of $y(x)$ gives the continuous block method (11) with the coefficients of $f_{n+j}$ given as

$$\sigma_0 = \frac{-1}{360}\left(3t^5 - 30t^4 + 110t^3 - 180t^2\right)$$

$$\sigma_1 = \frac{1}{120}\left(3t^5 - 25t^4 + 60t^3\right)$$

$$\sigma_2 = \frac{-1}{120}\left(3t^5 - 20t^4 + 30t^3\right)$$

$$\sigma_3 = \frac{1}{360}\left(3t^5 - 15t^4 + 20t^3\right)$$

Where $t = \frac{x - x_n}{h}$.

Evaluating the continuous block formula and it derivatives at $t = 1(1)3$, gives the discrete block formula of the form (2), where

$$e = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \ A^{(0)} = 6 \times 6 \text{ identity matrix,}$$

$$d = \begin{bmatrix} \frac{97}{369} & \frac{28}{45} & \frac{39}{40} & \frac{3}{8} & \frac{1}{3} & \frac{3}{8} \end{bmatrix}^T, \ b = \begin{bmatrix} \frac{19}{60} & \frac{22}{15} & \frac{27}{10} & \frac{19}{24} & \frac{4}{3} & \frac{9}{8} \\ \frac{-13}{120} & \frac{-2}{15} & \frac{27}{40} & \frac{-5}{24} & \frac{1}{3} & \frac{9}{8} \\ \frac{1}{45} & \frac{2}{45} & \frac{3}{45} & \frac{1}{24} & 0 & \frac{3}{8} \end{bmatrix}^T$$

It must be noted that we recover the continuous discrete scheme of Awoyemi et al[4] from the continuous block formula.

**3.0 Analysis of the method**

For the analysis of this method which include the order, consistency, zero stability and region of absolute stability, readers are referred to Awoyemi et al.[4]

**4.0 Numerical Examples**

**Problem 1:** Consider the non linear initial value problem

$$y''-x(y')^2=0 \quad y(0)=1, y'(0)=\frac{1}{2}$$

Exact solution: $y(x)=1+\frac{1}{2}In\left(\frac{2+x}{2-x}\right)$

Awoyemi et al.[4] solved this problem using step size of $\frac{1}{320}$. Despite the high step size of 0.01 we used, our method gives better approximation when compare with their result as shown in Table 1

**Problem 2:** Consider a highly oscillatory test problem

$$y''+\lambda^2 y=0 \quad y(0)=1, y'(0)=2$$

Exact solution: $y(x)=\cos 2x+\sin 2x$

We solved this problem for $h=0.01$. The result in Table 2 shows that our result is better that the result of Okunuga[11], who used $h=0.01$

ENR is Error in the new result

ENA is Error in Awoyemi et al.[5]

ENO is Error in Okunuga[12]

Table 1, Result of problem 1 with $h=0.01$

| $x$ | Exact result | Computed result | ENR | ENA |
|---|---|---|---|---|
| 0.1 | 1.050041729278490 | 1.0500417292786164 | 1.2501E-13 | 6.5501E-11 |
| 0.2 | 1.100335347731075 | 1.1003353477320403 | 9.6478E-13 | 5.4803E-10 |
| 0.3 | 1.151140435936466 | 1.1511404359397703 | 3.3035E-12 | 1.9256E-09 |
| 0.4 | 1.202732554054081 | 1.2027325540622877 | 8.2054E-12 | 4.8029E-09 |

| | | | | |
|---|---|---|---|---|
| 0.5 | 1.255412811882994 | 1.2554128119000487 | 1.7053E-11 | 1.0006E-08 |
| 0.6 | 1.309519604203111 | 1.3095196042350661 | 3.1954E-11 | 1.8727E-08 |
| 0.7 | 1.365443754271396 | 1.3654437543277900 | 5.6393E-11 | 3.2746E-08 |
| 0.8 | 1.423648930193603 | 1.4236489302892519 | 9.5649E-11 | 5.3969E-08 |
| 0.9 | 1.484700278594054 | 1.4847002787527916 | 1.5873E-10 | 8.8001E-08 |
| 1.0 | 1.549306144334058 | 1.5493061445961294 | 2.6207E-10 | 1.4353E-07 |

Table 2, Result of Problem 2 with $h = 0.01$

| $x$ | Exact result | Computed result | ENR | ENO |
|---|---|---|---|---|
| 0.01 | 1.01979867335991 | 1.019786733589537 | 9.5723E-13 | 2.6577E-11 |
| 0.02 | 1.03918944084761 | 1.039189440845429 | 2.1862E-12 | 8.1761E-10 |
| 0.03 | 1.05816454641465 | 1.0581645464109521 | 3.6965E-12 | 6.4146E-09 |
| 0.04 | 1.07671640027179 | 1.0767164002646032 | 7.1889E-12 | 6.7071E-09 |
| 0.05 | 1.09483758192485 | 1.0948375819138880 | 1.0965E-11 | 7.1209E-09 |
| 0.06 | 1.11252084314279 | 1.1125208431277500 | 1.5035E-11 | 7.6530E-09 |
| 0.07 | 1.12975911085687 | 1.1297591108356890 | 2.1184E-11 | 8.3601E-09 |
| 0.08 | 1.14654548998987 | 1.1465454899622458 | 2.7627E-11 | 9.0592E-09 |
| 0.09 | 1.16287326621395 | 1.1628732661795758 | 3.4369E-11 | 9.9268E-09 |
| 0.10 | 1.17873590863634 | 1.17897359085930247 | 4.3278E-11 | 1.0899E-08 |

## 5.0 Conclusion

We have proposed a non self starting continuous block method in this paper. The continuous block method enable us to evaluate a given problem at all the points within the interval of integration without starting the block all over. This property enables us to understand the behaviour of a dynamical system at any given point within the interval of integration. It had been shown from the numerical examples that non self starting method gives better approximation than the self starting method. We therefore recommend this method when solving second order initial value problems of ordinary differential equation.

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