The Reissner – Sagoci Problem for Ogden Solid.

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ABSTRACT

Finite deformation of the dynamic Reissner-Sagoci problem for a solid cylinder of Ogden material deforming under torsion and extension is analyzed for stresses and displacements. The boundary value problem resulted into a non-linear second order partial differential equation for the determination of displacements. An analytic solution of this is sought using the method of simple waves reduction, which in turn gave rise to eigenvalue – eigenvector problem. A closed form solution is provided for the determination of stresses and displacements at any cross-section.

Keywords: Deformation, incompressible, torsion, simple waves

1. INTRODUCTION

Reissner formulated the first problem of torsional vibration of the elastic half space in 1937. Seven years later solution was provided for the problem by Reissner and Sagoci [1]. Since then innvestigation has continued to centre on the static version. Many authors [2-18] have presented several variations of solutions of the Reissiner-Sagoci problem. However, not so much have been done for the dynamic version. This is expected, since the resulting equations are usually very highly non linear. The first attempt was made in

1944 [1] and the form of the solution was very unwieldy. Bycroft [19] in 1956 used Busbridge's analysis to obtain approximate solutions of a number of the dynamic variations of Reissner-Sagoci problem. Very recently Erumaka [20] presented an analytical solution of the dynamic Reisnner-Sagoci problem as an integral equation involving hypergeometric series at intermediate level.

This paper considers one of such dynamic cases of the Reissner-Sagoci problem for a cylindrical solid of an Ogden material which is forced to vibrate as a result of torsion applied at the top surface and simple extension on the longitudinal and lateral surfaces.

2. **Field Equations**

Let the cylindrical section of the half space be denoted by

$\Omega_0 = \{(r, \theta, z) : o \leq r \leq a, 0 \leq \theta \leq 2\pi, 0 \leq z \leq h\}$

in the undeformed configuration. The deformation which takes the point (r, θ, z) of the undeformed configuration to the point (R, ϕ, Z) of the deformed configuration $\Omega = \{R, \phi, Z)\}$ is given by

$$R = \lambda^{-1/2} r,\ \phi = \theta + w(z, t),\ Z = \lambda z \tag{2.1}$$

where λ is a positive constant

The deformation gradient tensor $\overline{F}$ is given by

$$\overline{F} = \begin{bmatrix} \lambda^{-1/2} & 0 & 0 \\ 0 & \lambda^{-1/2} & w_z r \lambda^{-1/2} \\ 0 & 0 & \lambda \end{bmatrix} \tag{2.2}$$

where w_z = partial derivative of w(z, t) with respect to z

The left Cauchy-Green deformation tensor

$$\overline{B} = \overline{F}\,\overline{F}^T$$

where $\overline{F}^T$ = transpose of $\overline{F}$, is given by

$$\overline{B} = \begin{bmatrix} \lambda^{-1} & 0 & 0 \\ 0 & \lambda^{-1}|\Delta|^2 & w_z r\lambda^{1/2} \\ 0 & w_z r\lambda^{1/2} & \lambda^2 \end{bmatrix} \tag{2.3}$$

where $|\Delta|^2 = 1 + w_z^2 r^2$

The strain invariants are

$$I_1 = \lambda^2 + \lambda^{-1}(1 + |\Delta|^2)$$

$$I_2 = 2\lambda + \lambda^{-1}|\Delta|^2$$

$$I_3 = 1 \tag{2.4}$$

Let us consider the Ogden Solid which is characterized by the strain energy density function of the form

$$W = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\alpha_{mn}\ (I_1 - 3)^m(I_2 - 3)^n \tag{2.5}$$

where W = 0 for m ≠ n and α_{mn} are constants. In this paper we consider the case m = n = 1. For this case the stress tensor $\overline{\tau}$ for an incompressible solid is given by

$$\overline{\tau} = -PI + 2W_1\overline{B} - 2W_2\overline{B}^{-1} \tag{2.6}$$

where ρ is the hydrostatic pressure and I is the unit tensor. Using (2.4), (2.5) in (2.6) we have the non-zero components of stress as

$$\tau_{rr} = \rho + 2\lambda^{-1}\alpha\,(I_2 - 3) - 2\alpha(I_1 - 3)\lambda$$

$$\tau_{\theta\theta} = -\rho = 2\alpha\,|\Delta|^2\lambda^{-1}(I_1 - 3) - 2\lambda\alpha\,(I_2 - 3) \tag{2.7}$$

$$\tau_{r\theta} = 2\lambda^{-1/2}\alpha\,(I_2 - 3)w_z r - 2\lambda^{1/2}\alpha\,(I_1 - 3)w_z r$$

$$\tau_{zz} = -\rho + 2\lambda^2\alpha\,(I_2 - 3) - 2\lambda^{-2}\,(I_1 - 3)|\Delta|^2$$

Where α_{11} is replaced by α for convinience

Using (2.7) the non trivial equation of motion is

$$\frac{\partial \tau_{\theta z}}{\partial z} = \rho\frac{\partial^2 u}{\partial t^2}$$

which reduces to

$$\left(\frac{\beta}{\rho} + \frac{\xi}{p}\,u_z^2\right) u_{zz} = u_{tt} \tag{2.8}$$

where β and ξ are constants

3. THE BOUNDARY VALUE PROBLEM

We consider a problem in which a cylindrical section of radius r is forced to rotate about its central axis

With reference to Fig. 1 we need to solve equation (2.8) subject to the boundary conditions

$$u(z, 0) = h(z) \tag{3.1}$$

where we have assumed that u = 0 as $(r^2 + z^2)^{1/2} \to \infty$

We first re-write equation (2.8) as

$$u_{tt} = (n + m^2 u_z^{\,2}) u_{zz} \tag{3.2}$$

where $n = \dfrac{\beta}{\rho}$ and $m = \dfrac{\xi}{\rho}$

Now by setting $u_1 = u_z$ and $u_2 = u_t$, equation (3.2) reduces to the system

$$\frac{\partial}{\partial t}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = A(u)\frac{\partial}{\partial z}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \tag{3.3}$$

$$\text{where } A(u) = \begin{pmatrix} 0 & 1 \\ n + m^2 u_1^2 & 0 \end{pmatrix} \tag{3.4}$$

Now if we assume that there exists a scalar function θ of z and t such that

$$\underline{u} = \underline{f}(\theta) \tag{3.5}$$

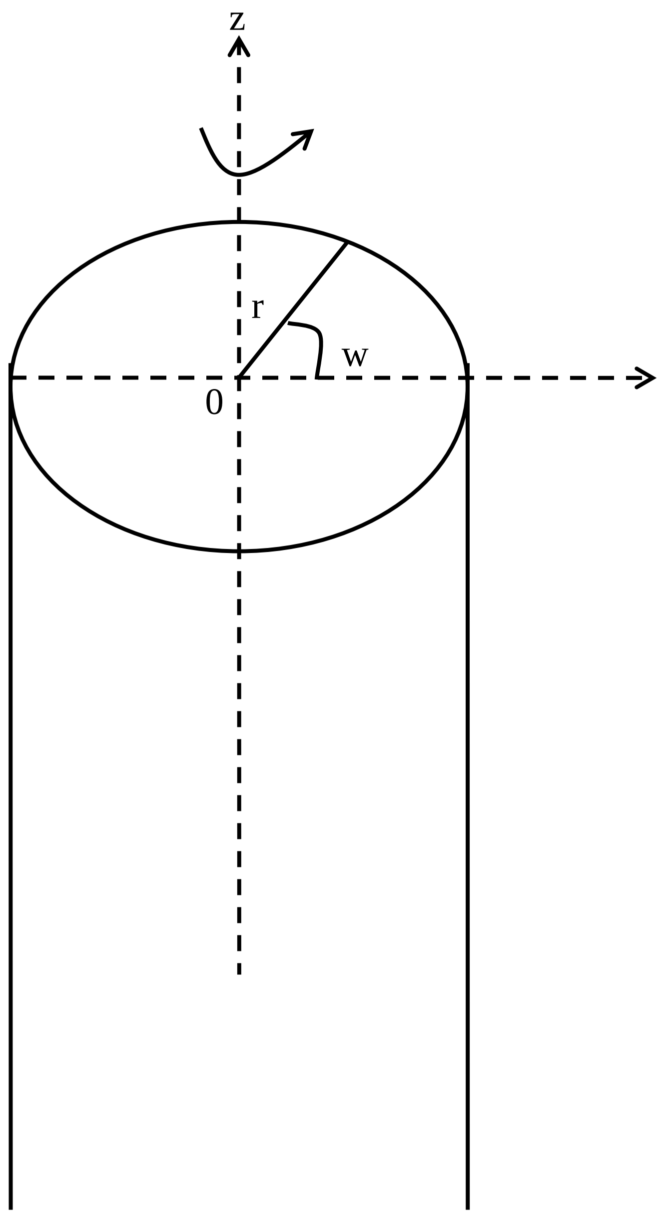

Fig. 1: A cylindrical section forced to rotate about its central axis, z by a twist w(z, t)

then equation (3.3) can be expressed as

$$\begin{pmatrix} f_1' \\ f_2' \end{pmatrix} \frac{\theta_t}{\theta_z} = A(f_1, f_2) \begin{pmatrix} f_1' \\ f_2' \end{pmatrix} \qquad (3.5)$$

We see that equation (3.5) is an eigenvalue-eigenvector problem in which

$$r = \pm (n + m^2 f_1^2)^{1/2} \qquad (3.6)$$

are the eigenvalues and (f'_1, f'_2) the corresponding eigenvectors.

It is easy to see that this leads to the equation

$$f'_2 = \pm (n + m^2 f_1^2)^{1/2} f'_1 \qquad (3.7)$$

as the relation between the two eigenvectors. Integrating equation (3.7) we obtain

$$f_2 = \pm (\square_1 + \square_2) \qquad (3.8)$$

$$\square_1 = \frac{\sqrt{n}}{2}\left[1 + \frac{m^2 f_1^2}{n}\right]^{1/2} \qquad (3.9)$$

$$\square_2 = \frac{\sqrt{n}}{2m} \mathrm{In}\left[\left(1 + \frac{m^2 f_1^2}{n}\right)\right]^{1/2} + \frac{m f_1}{\sqrt{n}} \qquad (3.10)$$

Equation (3.8) is a first order non-linear partial differential equation of the form

$$G(z, t, u, p, q) = 0 \qquad (3.11)$$

where $p = u_z$, $q = u_t$

We use the method of characteristics with the initial curve as

$$\begin{cases} z_o = s \\ t_o = o \\ u_o = h(s) \end{cases} \tag{3.12}$$

and the solution of the boundary value problem becomes

$$u(z, t) = \frac{1}{2}\left\{\frac{mh'(z)}{\left[1+\left(\frac{mh'(z)}{\sqrt{n}}\right)^2\right]^{1/2}} + \frac{nh'(z)}{m}\left[1+\left(\frac{mh'(z)}{\sqrt{n}}\right)^2\right]^{1/2}\right.$$

$$\left. + \frac{nh'(z)}{\sqrt{n}\left[1+\left(\frac{mh'(z)}{\sqrt{n}}\right)^2\right]^{1/2}} + \frac{n}{m}\right\}t + czt + h(z) \tag{3.13}$$

The corresponding stress components are got by using equation (3.13) in equation (2.7)

4. SUMMARY AND CONCLUSION

By employing method of reduction to simple wave forms we have succeeded in presenting an analytic solution to the dynamic Reissner-Sagoci problem for a typical incompressible solid as opposed to the parametric solution of [1] and hypergeometric series presentation of [20]. A careful substitution and simplification gives the stress components under the considered mode of deformation in the radial direction as

$$\tau_{rr} = K + 2\alpha \left(\frac{1-\lambda^3}{\lambda^3}\right) u_z^2$$

where K is a constant. We observe that the stress in the radial direction is a constant if $\lambda = 1$, which agrees with the result obtained in [20]

The hoop stress $\tau_{\theta\theta}$ gives on simplification

$$\tau_{\theta\theta} = 2\alpha \left(\lambda - \frac{2}{\lambda} + \frac{u_z^2}{\lambda}\right)\left(1 + u_z^2\right) - \rho - 4\alpha\lambda^2$$

where u is as derived in (3.13)

Also τ_{zz} is given by

$$\tau_{zz} = 2\alpha \left(\frac{3}{\lambda^2} - 1 - 2\lambda^3 - \frac{u_z^2}{\lambda^3}\right)\left(1 + u_z^2\right) + 2\alpha u_z^2 + 4\lambda^4 - 4\alpha\lambda^2 - \rho$$

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