Tomonaga-Luttinger Liquid for Collective Motion of Fermions: a Basic Theory of One-Dimensional Correlated Electrons.

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ABSTRACT

In this paper, we discuss fundamental properties of the Tomonaga-Luttinger liquid in comparison with the Fermi liquid, and explain how (1+1) dimensional conformal field theory can be used to describe universal properties of the Luttinger liquid. Some examples of recent experimental observations are also addressed.

Keyworks: Tomonaga-Luttinger Liquid, Fermi Liquid, electron correlation.

1. INTRODUCTION

The principal problem of physics is to determine how bodies behave when they interact. In condensed matter physics, electron correlations, which result from mutual electron-electron interactions, give rise to a wide variety of interesting phenomena such as magnetism, superconductivity, etc. One of the central issues in condensed matter theory has been how to treat the problem of such electron correlations, which is a typical many-body problem. Recently, electron correlations in low dimensions have attracted much attention. This is because large quantum fluctuations intrinsic in low dimensions, together with strong electron correlations, result in new and unusual quantum states that have not been observed in three dimensional systems. A well known example is the quantum Hall effect, discovered February 5, 1980, which occurs in a two-dimensional electron system with strong magnetic fields. The new and unusual quantum liquid state realized in this system gives the Hall conductance quantized with accuracy of 10^{-8} [1].

A fundamental question that comes to mind is: what happens in a one-dimensional interacting many-electron system, where the effect of quantum fluctuations should be very strong. One naturally expects that anomalous quantum phenomena, which are quite different from those in three-dimensional electron systems, should emerge reflecting electron correlations accompanied by large quantum fluctuations. Remarkably, this fundamental problem was posed by Sin-itiro Tomonaga, in 1950 [2]. He explained that collective motion of electrons, instead of individual electron motion, plays a key role in physics of one-dimensional correlated electron systems.

Tomonaga's ground-breaking work on one-dimensional interacting electrons was titled "Remarks on Bloch's Method of Sound Wave Applied to Many-Fermion Problems". This paper [2] proposed general approach to treat collective motion of fermions in one spatial dimension in terms of harmonic oscillator description, which is now known as the bosonization method [3, 4]. This work has then motivated vast theoretical investigations, and has established the basic theory of one-dimensional many-body systems, the so-called Tomonaga-Luttinger (TL) liquid [2, 5, 6]. Interestingly, it turns out that this theory covers not only fermions but also a wide variety of quantum systems including interacting bosons and spins. At the time the paper was published, there were very few experiments on one-dimensional quantum systems. Today, we have a number of correlated electron systems in which clear evidence for anomalous TL liquid properties has been found experimentally [7, 8, 9], and such experimental advances provide exciting opportunities to test universal TL liquid and quantum critical phenomena in low dimensional many-body system [10].

The discovery of high-T_c superconductivity in April, 1986 [11, 12, 13], was another event that aroused renewed interest in the TL liquid. This posed a fundamental question in condensed matter physics, i.e. how the strongly correlated electrons behave in low dimensions. Infact, the fundamental question was focused on whether the Fermi-liquid (FL) theory, the basic idea in three-dimensional interacting fermions, persists in low-dimensional strongly correlated systems [14, 15, 16, 17]. This triggered further intensive studies on one-dimensional correlated electrons that indeed exhibit anomalous non-Fermi liquid properties – Tomonaga-Luttinger liquid. Since the original approach initiated by Tomonaga was based on a weak-coupling theory, the main problem was how to treat the TL liquid in the strong correlation system, which has been attacked by various numerical and analytical methods early in 1990s. Among others, conformal field theory (CFT) [18, 19, 20], which has been advanced in 1980s in connection with superstring theory, and condensed matter physics in two-dimension, has provided us with a beautiful method to describe the TL liquid in the strong coupling system from the viewpoint of critical phenomena in 1+1 space-time dimensions.

In this paper, we briefly discuss the TL liquid in comparison with the FL, and outline how CFT techniques can be applied to the TL liquid. Recent advances in experiments are also mentioned [15, 21].

2. COMPARISM OF TOMONAGA-LUTTINGER LIQUID WITH FERMI LIQUID

The ground-breaking Tomonaga model is spinless fermion model which has a linear dispersion relation with a finite band D (Fig.1) [2]. Tomonaga treated collective excitations of the model in terms of boson operators. That is, he described the system in term of density field $\rho(x)$, which obeys the boson commutation relation in the wide band limit instead of wave field $\psi(x)$.

$$\rho_n = \sum_{\bar{n}} \psi^*_{\bar{n}-\frac{n}{2}} \psi_{\bar{n}+\frac{n}{2}} \tag{1}$$

ρ_n is seperated into two parts ρ_n^+ and ρ_n^- by means of

$$\left.\begin{aligned} \rho_n^+ &= \sum_{\bar{n}>0} \psi^*_{\bar{n}-\frac{n}{2}} \psi_{\bar{n}+\frac{n}{2}} \\ \rho_n^+ &= \sum_{\bar{n}<0} \psi^*_{\bar{n}-\frac{n}{2}} \psi_{\bar{n}+\frac{n}{2}} \end{aligned}\right\} \tag{2}$$

We have evidently

$$\rho_n = \rho_n^+ + \rho_n^- \tag{3}$$

where, ψ_n *and* $\psi^*{}_n$ are the quantized wave functions describing the assembly, $\bar{n}$ takes integral values when n is even and half-odd integral values when n is odd. The Tomonaga model with $D\to\infty$ is referred to as the TL model [5]. When more realistic models which do not have the linear dispersion are considered, we focus on the low-energy physics, and linearize the dispersion around the Fermi level, as shown in Fig. 1. We then arrive at the TL model as an effective low-energy theory, for which we can apply standard bosonization techniques [22].

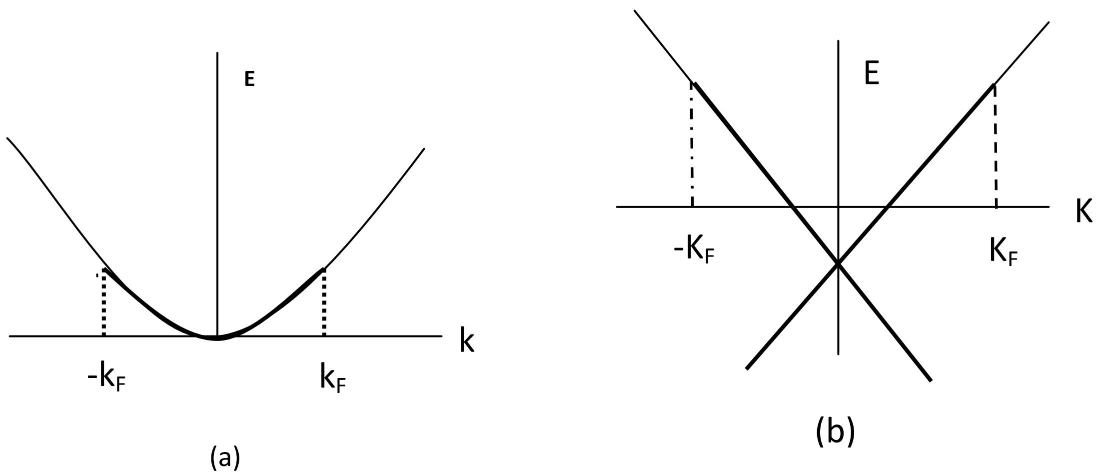

Fig.1: (a) one-dimensional fermion model: fermions are filled up to the Fermi wave number k_F , (b) Tomonaga-Luttinger model with linear dispersion. The original Tomonaga model has a finite bandwidth D, while in the TL model, $D\to\infty$.

We now compare the TL liquid with the FL. The universal role of the FL is well established to describe low-energy properties of correlated electrons in three dimensions: even in the presence of interactions the electrons can retain original fermionic properties although the electrons are renormalized to form quasi-particles with heavy mass. However, this is not necessarily valid in low-dimensions (for low-dimensional systems, as conventional FL theory fails due to strong correlation effects) [23, 24]. In particular, in one dimension, large quantum fluctuations make the quasi-particle states completely unstable, and the low-energy physics is described by the TL liquid. It's worth noting that, static quantities such as the spin susceptibility and the specific heat show similar properties in the TL liquid and the FL. For example, the static spin susceptibility is constant at low temperatures in both cases (Pauli paramagnetism). Therefore, we cannot distinguish these two liquids in terms of such static quantities. The most remarkable behavior, which clearly distinguishes the TL liquid from the FL, is the anomalous power-law dependence of various correlation functions in the low-energy region. For example, the density-density correlation function between two distant positions *x* and *x'* exhibits the power-law dependence in the asymptotic region,

$$< \rho(x)\rho(x') > \sim e^{2ik_F(x-x')}(x-x')^{-\alpha} \tag{4}$$

The remarkable point is that the critical exponent α changes continuously depending on the strength of the interaction between particles. This is quite distinguished from the FL where the critical exponent is fixed to an integer. This implies that the quasi-particle (electron) picture in the FL does not hold in the TL liquid, and the low-energy properties in the TL liquid are described by critical phenomena of collective modes.

Another example of correlation functions that show similar power-law dependence is the density of states (DOS) shown in Fig.2 for the TL liquid in comparison with the FL. Note that the density of states is obtained via a Fourier transform of the one-electron Green function, so that it should show power-law behavior in the TL case. In contrast to the FL, which has the finite DOS at the FL, the DOS for the TL liquid vanishes like $|E—E_F|^{\beta}$ (for repulsive interactions). This quantity is thus used to characterize the TL liquid, which can be measured by photoemission experiments, as we will see below.

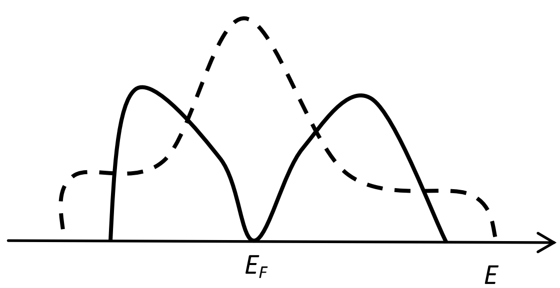

Fig .2: Density of states for the FL (broken line) and the TL liquid (solid line) as a function of the energy E. In the TL liquid, there appears the $|E—E_F|^{\beta}$ *dependence around the Fermi level* E_F [25].

As mentioned above, the standard approach of Tomonaga can be applied to weak-coupling systems. Since we are now interested in the strong correlation system, the problem is how to describe TL liquid properties in the whole coupling system including the crossover from the weak to strong coupling limit. This problem was addressed by powerful numerical methods and was resolved fairly [26, 27].

Nevertheless, generally speaking, it is quite difficult to obtain the critical exponents numerically. In this link, we wish to bring to mind that the exact solution via the Bethe ansatz [28, 29, 30] is available for a certain class of correlated models in one-dimension. A familiar and important example is the Hubbard model [31] which describes correlated electrons on the lattice with the influence of on-site repulsive interactions. Although static quantities were calculated exactly by the Bethe ansatz method [27], the calculation of correlation functions, which is crucial to discuss the TL liquid, has been an open problem to be solved. This is why the TL properties in the strong coupling system have not been clarified for long time in spite of the existence of the exact solution. However, this problem can be resolved by combining CFT with the exact solution [6, 25, 32, 33].

3. CONFORMAL FIELD THEORY

CFT describes universal behavior of macroscopic properties in (1+1) dimensional critical systems based on the representation theory of underlying infinite-dimensional symmetry [19, 20, 33]. Conformal symmetry is symmetry under scale invariance and under the special conformal transformations having the following relations [18].

$$\left.\begin{aligned} &[P_\mu, P_\nu] = 0, \\ &[D, K_\mu] = -K_\mu, \\ &[D, P_\mu] = P_\mu, \\ &[K_\mu, K_\nu] = 0, \\ &[K_\mu, P_\nu] = \eta_{\mu\nu} D - i M_{\mu\nu}, \end{aligned}\right\} \qquad (5)$$

where P generates translations, D generates scaling transformations as a scalar and K_μ generates the special conformal transformations as a covariant vector under Lorentz transformation. According to experiments, in higher dimensions the physical quantities associated with the phase transition vary as $|T\text{-}T_c|^{\alpha}$ when the critical temperature T_c is approached. The critical exponent α takes universal values irrespective of the material under consideration [34]. But phase transition does not occur at finite temperatures in one spatial dimension, because long-range correlations are destroyed by thermal fluctuations [35], and hence, the correlation functions decay exponentially at long distance [24]. However, the criticality can be realized at zero temperature, where the correlation length becomes infinite, giving rise to the power-law behavior in long-distance or long-time asymptotics of correlation functions as in the TL liquid [24, 35]. In such critical phenomena, there appears the global scale invariance: even if we change the length scale uniformly, physics should not be changed. Conformal invariance is a local version of the scale invariance shown schematically in Fig. 3,

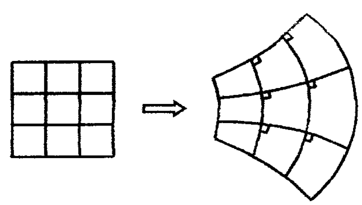

Fig 3: Conformal transformation in (1+1) dimensions [18].

which is expected to hold for any critical systems with short range interactions. In particular, such conformal invariance plays a remarkable role in (1+1) space-time dimensions, since it contains an infinite number of generators. The corresponding generator L_n (n = 0, ± 1, ± 2...) of conformal transformations thus forms the infinite-dimensional Lie algebra, the so-called Virasoro algebra [33, 34],

$$[L_m, L_n] = (m-n) L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}, \tag{6}$$

which provides a powerful method to analyze critical phenomena in (1+1) dimensions. The Virasoro algebra is characterized by the central charge c in the second term. The important role of c is the classification of the universality class of critical phenomena [24]: e.g. the TL liquid is specified by $c = 1$. There is another important parameter in CFT, the scaling dimension x_s (or conformal dimension), which represents the dimension of a given field under the global scale transformation. Note that x_s itself determines the critical exponent of correlation functions. Therefore, if we obtain these parameters microscopically, the corresponding critical phenomena can be described in terms of CFT.

A practical way to compute the conformal parameters is the finite-size scaling technique developed in CFT [36]. For example, let us consider a one-dimensional fermion system with linear size L under periodic boundary conditions. The scaling dimension x_s appears in the low-energy excitation spectrum ΔE in the universal form,

$$\Delta E \sim \frac{2\pi v_s x_s}{L} \tag{7}$$

where v_s is the velocity of collective motion of fermions. This important formula enables us to compute the correlation exponent x_s from the excitation spectrum. Note that the latter can be calculated via the Bethe ansatz, so that we can obtain the correlation exponent exactly.

We now ask whether the above idea in CFT can be immediately applied to one-dimensional electron systems. It is known that electrons carry low-energy charge and spin excitations, but with different velocities. At first glance, the system may not be compatible with CFT. However, charge and spin degrees of freedom are separated in the continuum limit (spin-charge separation), each of which can be described by distinct conformal field theories. Therefore, by analyzing the spectrum of the exactly solvable model such as the Hubbard model in terms of CFT, we can obtain the critical exponent of the correlation functions. This technique was successfully applied by Norio, K. [25], Holger, F. [31] and Schulz, H. J. [32] to obtain the critical exponents of the Hubbard model exactly. In this way, CFT enables us to describe the TL liquid in terms of infinite symmetry of the underlying (1+1) critical phenomena, and gives us a practical way to compute various correlation exponents.

4. EXPERIMENTAL OBSERVATIONS

There are many systems which can be regarded as one-dimensional electron systems. Typical examples are quantum wires, carbon nanotubes [1, 16, 37], edge states of quantum Hall systems [38, 39, 40, 41], tetrathiofulvalinium-tetracyano-quinodimethane (TTF-TCNQ) [42, 43, 44].

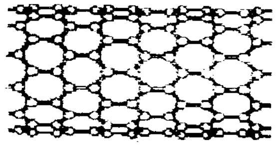

Fig. 4: Schematic structure of a carbon nanotube [17].

In these systems, anomalous power-law behavior of correlation functions has been observed in the conductance, the photoemission spectrum, the NMR relaxation rate, etc. Such anomalies appear explicitly in the dependence on temperature, frequency, voltage, magnetic field, etc. Also, copper-oxide chain/ladder materials, which have been synthesized in connection with high-T_c superconducting copper oxides, give another class of the TL liquid, in which spin excitations exhibit TL liquid properties in magnetic fields. More recently, cold fermions and bosons trapped in a one-dimensional optical lattice provide a new research area of the TL liquid. Here we quote a recent observation of the photoemission spectrum for carbon nanotubes. A carbon nanotube is made of graphite sheet, which is an ideal one-dimensional material with diameter 1-30nm and length 10μm [22] (Fig. 4). The carbon nanotube is either metallic or semiconducting according to how to roll up the graphite to make a tube. Anomalous power-law behavior has already been observed in various transport quantities and dynamical response functions for metallic cases [7, 23, 44].

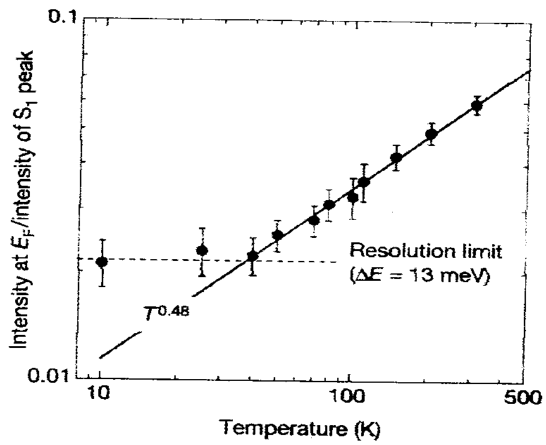

Fig.5: High-resolution photoemission spectra of the singlewall carbon nanotube near the Fermi level measured at different temperatures, T= 10 K, 40 K, 70K, 150 K and 310 K with an energy resolution of 13 meV (Hiroyoshi, I. et al. [44]).

As an example, the temperature-dependence of the photoemission intensity at the Fermi level observed by Hiroyoshi *et al.* is shown in Fig. 5 [44]. Since an electron is removed from the system in the photoemission process, this experiment is used to study the spectrum of the one-electron Green function *(see Fig. 2)*. It is seen that the photoemission intensity exhibits clear power-law temperature dependence with the anomalous exponent 0.46. This behavior has also been confirmed by the complementary observation of the energy dependence of the intensity at a given temperature, from which the exponent 0.48, quite consistent with its temperature dependence, is obtained.

Another example is the power-law behaviour of the DOS $\rho(\omega)\ |\omega|^{\alpha}\ (\omega = E_F - E)$, for the optical and photoemission evidence for a Tomonaga-Luttinger liquid in the Bechgaard salts experiment [45]. The exponent in this expression reflects the nature and strength of the interaction. Fig. 6, compares the typical DOS near the chemical potential and at T = 0 for a FL and a TL liquid (with $\alpha > 1$).

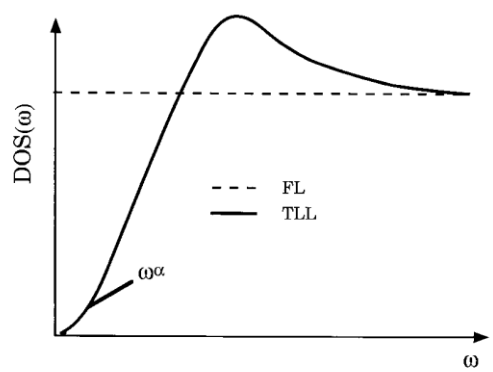

Fig. 6: Comparison between the Fermi-liquid and Tomonaga-Luttinger liquid density of states [46].

5. CONCLUSION

The concept of TL liquid, initiated by Tomonaga in 1950, has established an exciting field in condensed matter physics. Extensive studies based on bosonization approaches, numerical methods, CFT analyses, etc, have explained its anomalous properties in the whole constraint system from weak to strong couplings. We can see a beautiful mathematical structure in this framework that governs universal behaviour observed in a wide variety of real materials in one-dimension. A concrete study of the TL liquid will be further advanced in the future in the fields of nanoscale systems, cold atoms in optical lattices, etc.

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