A non-uniform Spherical Stuart vortex

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Abstract

A spherical vortex comprising of a uniform steady flow located at the centre of circulation and a non-uniform stream emanating from the uniform vortex is presented in this paper as an example of exact description of Stuart vortices on the surface of a sphere. The vortex streams obtained results from a solution of a generalized set of partial differential equations presented recently in the literature generalizing Stuart vortices to the surface of a sphere.

1. Introduction

In 1967 Stuart came up with the exact solution of the steady two-dimensional Euler equation which is now well known in the field of fluid dynamics as "Stuart vortices" [1], Stuart solution is a model of the free shear layer where the vorticity stream function relationship is given by

$$\omega = -\nabla^2\psi = ce^{-2\psi} \tag{1}$$

Where c is a constant, ω is the vorticity and ψ is the stream function.

This is also known as the Liouville equation. Stuart was able to make a useful contribution in Fluid dynamics by accurately discussing solutions of equation (1) which on the long run has been useful to astronomers and geophysicist. Stuart solution is one of the exact solutions of the planar Euler equations. Most of the other solutions are weak solutions involving vortex patches, see for example [2]. [3] was one of the first to formulate the mathematical study of point vortex dynamics on a sphere .

To anticipate vertical structures on planetary scale where curvature effects play an important role, (1) must be generalized to the surface of the sphere.

A major drawback in generalizing (1) to a sphere arises from the fact that spherical surface is closed. Therefore, if a solid boundary is present such as in planetary motions, any vorticity distribution on the surface must satisfy the Gauss condition.

$$\int_s \omega d\sigma = 0 \tag{2}$$

That is the integral of the vorticity ω over the entire spherical surface s must be taken to be zero. This leads to

$$\omega = -\nabla_\Sigma^2 \psi = ce^{d\psi} + \frac{2}{d} \tag{3}$$

Where ∇_Σ^2 is the Laplace Beltrami, c, d are constants

2 DERIVATION OF THE GOVERNING EQUATIONS

Consider the spherical Co-ordinate system, where

$$x = \sin\phi\cos\theta$$
$$y = \sin\phi\sin\theta \tag{4}$$
$$z = \cos\theta$$

and $0 \le \phi < \pi \quad , \quad 0 \le \theta < 2\pi$ (5)

We assume a unit sphere hence $r = 1$, in (4) transforming ∇_Σ^2 to spherical co-ordinates we get

$$\nabla_\Sigma^2 = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial}{\partial\theta}) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} \tag{6}$$

Consider

$$U = (u, v, o) \tag{7}$$

(7) follows from axial symmetry i.e. the incompressible nature of the flow allow the introduction of a scalar stream function $\psi(\theta, \phi)$.

We have that

$$u = -\frac{\partial \psi}{\partial \theta} \quad , \quad v = \frac{1}{\sin\theta}\frac{\partial \psi}{\partial \theta} \tag{8}$$

$$\omega = -\nabla_{\Sigma}^{2}\psi \tag{9}$$

In the steady case, the material conservation of vorticity is expressed as

$$\left(\frac{u}{\sin\theta}\frac{\partial}{\partial \phi} + v\frac{\partial}{\partial \theta}\right)\omega = 0 \tag{10}$$

Substituting (8) into (10) we obtain

$$\frac{1}{\sin\theta}\left(-\frac{\partial \psi}{\partial \theta}\frac{\partial \omega}{\partial \phi} + \frac{\partial \psi}{\partial \phi}\frac{\partial \omega}{\partial \theta}\right) = 0 \tag{11}$$

This implies that for a solution of (11) to be obtained the vorticity ω must be a function of the stream function ,i.e

$$\omega = h(\psi) \tag{12}$$

for some differentiable function h.The choice of h determines the vorticity -stream function relation.Lets choose

$$\omega = -ce^{d\psi} \tag{13}$$

since we wish to extend the solution to the sphere then there must exist a global constraint on the vorticity distribution

(13) must be modified to satisfy the Gauss constraint (2) hence we invoke a constant say g.From (9) and (13) we obtain

$$\nabla_{\Sigma}^{2}\psi = ce^{d\psi} + g \tag{14}$$

Where g is a constant related to d in the special case as [1]

$$g = \frac{2}{d} \tag{15}$$

Where the special case corresponds to the standard Liouville equation.

(14) Is known as a modified liouville equation. We must transform (14) to a complex plane with the help of stereographic projection

3. STEREOGRAPHIC PROJECTION

Define

$$\zeta = re^{i\phi} \tag{16}$$

Where

$$r = \cot\frac{\theta}{2} \tag{17}$$

ζ=0 conforms to the south pole of the sphere with the relations given below

$$cos\theta = \frac{\zeta\overline{\zeta} - 1}{\zeta\overline{\zeta} + 1} \qquad \sin\theta = \frac{2 \times \sqrt{\zeta\overline{\zeta}}}{\zeta\overline{\zeta} + 1} \tag{18}$$

$$\left.\frac{\partial}{\partial\theta}\right\|_{\phi} = -\frac{\zeta}{\sin\theta}\left.\frac{\partial}{\partial\zeta}\right\|_{\overline{\zeta}} - \frac{\overline{\zeta}}{\sin\theta}\left.\frac{\partial}{\partial\overline{\zeta}}\right\|_{\zeta} \tag{19}$$

Similarly

$$\left.\frac{\partial}{\partial\phi}\right\|_{\theta} = i\zeta\left.\frac{\partial}{\partial\zeta}\right\|_{\overline{\zeta}} - i\overline{\zeta}\left.\frac{\partial}{\partial\overline{\zeta}}\right\|_{\zeta} \tag{20}$$

We are led to the following equation

$$\nabla_{\Sigma}^{2}\psi = (\zeta\bar{\zeta}+1)^{2}\psi_{\zeta\bar{\zeta}} \tag{21}$$

4. TRANSFORMATION OF THE DEPENDENT VARIABLE

We introduce a dependent variable

$$\phi(\zeta,\bar{\zeta}) \equiv \psi(\zeta,\bar{\zeta}) - \frac{2}{d}\log(1+\zeta\bar{\zeta}) \tag{22}$$

Differentiating (22) w.r.t to ζ and $\bar{\zeta}$ we obtain

$$\phi_{\zeta\bar{\zeta}} \equiv \psi_{\zeta\bar{\zeta}} - \frac{2}{d}\frac{1}{(1+\zeta\bar{\zeta})^{2}} \tag{23}$$

Which implies the following

$$\phi_{\zeta\bar{\zeta}} + \frac{2}{d}\frac{1}{(1+\zeta\bar{\zeta})^{2}} = ce^{d\phi} + \frac{g}{(1+\zeta\bar{\zeta})^{2}} \tag{24}$$

(24) can be written in the form

$$\phi_{\zeta\bar{\zeta}} = ce^{d\phi} + \frac{k}{(1+\zeta\bar{\zeta})^{2}} \tag{25}$$

Where

$$k = g - \frac{2}{d} \tag{26}$$

When $k = 0$ (25) gives a standard Liouville equation

$$\phi_{\zeta\bar{\zeta}} = ce^{d\phi} \tag{27}$$

Let

$$\Phi = \phi - k\log(1+\zeta\bar{\zeta}) \tag{28}$$

Hence

$$\Phi_{\zeta\bar{\zeta}} = \phi_{\zeta\bar{\zeta}} - \frac{k}{\left(1+\zeta\bar{\zeta}\right)^2} \tag{29}$$

Therefore (25) implies [4]

$$\Phi_{\zeta\bar{\zeta}} = c(1+\zeta\bar{\zeta})^{\gamma} e^{d\Phi} \tag{30}$$

Which is also a modified Liouville equation where

$$\gamma = dk \tag{31}$$

Changing notations for convenience

$$C = dc \tag{32}$$

Then

$$\Phi_{xy} = \frac{C}{d}(1+\zeta\bar{\zeta})^{\gamma} e^{d\Phi} \tag{33}$$

(33) Generalizes Stuart's solution to a sphere. The choice $g = \frac{2}{d}$ corresponds to $\gamma = 0$ and (33) becomes a standard Liouville equation. Explicit representation of the general solution in a closed form in this case is given in [1] and [5].We find the general solution to the 2.D Liouville equation, other choices of k and other combination of the parameter d and g corresponds to $\gamma \neq 0$. It is almost imposible to find explicit solution to (33) for $\gamma \neq 0$.To establish a connection between the general solution and vortex structures Crowdy sets c$= 1$ and hence $d < 0$.This implies that c and d must have opposite signs. This is generally not the case for $\gamma \neq 0$.In [1] it was an open mathematical question to find if other choices of g will also lead to solutions satisfying (2). Motivated by this we wish to examine the case for $\gamma = -3$ which corresponds to the choice $g = -\frac{1}{d}$ where d represent the strength of the vorticity everywhere on the sphere [1] .Solution to (33) for $\gamma = -3$ is given by

$$\Phi = \frac{1}{d}\log\left(\frac{1+\zeta\bar{\zeta}}{cd}\right) \tag{34}$$

Provided c and d have the same sign i.e. $cd > 0$. (34) also generates point vortices (when $\zeta\bar{\zeta} = -1$).One of the major difference in [1] and the solution presented herein is that $cd < 0$ in crowdys' solution while in (34) $cd > 0$ and the solutions tends to have slightly different geometry. We wish to examine the behavior of (34)

Diagrammatic Representation

Fig1

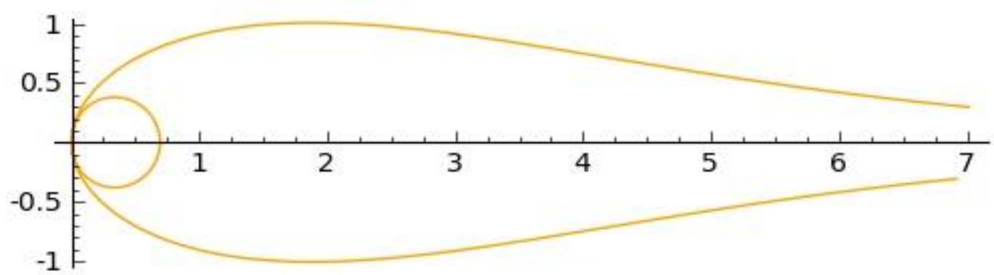

Fig (I) shows a diagrammatic representation of equation (34) in polar co-ordinate.A closer look at Fig(I) reveals a streamlined strut or an airfoil. There seem to be a singular point as $r \to \infty$. (34) describes a spherical vortex.

6. CONCLUSION

A spherical vortex comprising of a uniform steady flow located at the centre of circulation and a non-uniform stream emanating from the uniform vortex has been found as an example of exact description of Stuart vortices on the surface of a sphere. The vortex streams obtained results from a solution of a generalized set of partial differential equations presented recently in the literature generalizing Stuart vortices to the surface of a sphere. [4], In this paper we explore other choice of g which is $\gamma = -3$ which corresponds to $g = \frac{1}{d}$. A physical admissible Stuart vortex in this case is given as

$$\Phi = \frac{1}{d}\log\left(\frac{1 + \zeta\bar{\zeta}}{cd}\right)$$

The behavior and shape is known by plotting the above function.

References

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