SOLVING LINEAR AND NONLINEAR INITIAL VALUE PROBLEMS BY THE ADOMIAN DECOMPOSITION METHOD

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Abstract

In this paper, the Adomian Decomposition Method (ADM), a numerical method which gives the solution as a series is presented. *We have chosen to illustrate this method by solving first and second order IVPs. Some examples are solved to illustrate the efficiency of the method, comparison with exact solutions and Taylor series method is also given.*

Key words: Adomian Decomposition method, Adomian Polynomials, Initial value problems.

1. Introduction

The Adomian Decomposition method provides the solution as an infinite series [1] in which each term is determined and unlike other methods which are based on discretization principles ADM does not avoid some fundamental phenomena, it also avoids linearization and perturbation [2]. The method can effectively solve a large class of linear and nonlinear differential and integral equation [3], the method has been used to derive analytical solution for nonlinear ordinary differential equation [4]. It is a better and more accurate solution method for determining approximate or exact solution to IVPs.

2. Adomian Decomposition Method

Consider the IVP:

$$\left.\begin{array}{r} u' = f(t,u) \\ u(t_0) = u_0 \end{array}\right\} \qquad (1)$$

Equation (1) is written in an operator form as:

$$Lu = f(t,u) \qquad (2)$$

where the differential operator

$$L = \frac{d}{dt} \qquad (3)$$

and the inverse operator given by

$$L^{-1} = \int_0^t dt \qquad (4)$$

Applying L^{-1} on (2) and imposing the initial condition, we have

$$u(t) = u_0 + L^{-1} f(t,u) \qquad (5)$$

$f(t,u)$ is decomposed into $Ru + Nu$, where Ru is a linear differential operator and Nu is a nonlinear operator

Adomian decomposition method defines the solution $u(t)$ by the series

$$u(t) = \sum_{n=0}^{\infty} u_n(t) \qquad (6)$$

where the components $u_n(t)$ are usually determined recurrently by using the relation

$$\left.\begin{aligned} u_o &= F \\ u_{n+1} &= L^{-1}(Ru_n) - L^{-1}(Nu_n) \end{aligned}\right\} \qquad (7)$$

and F is the term arising from integrating the source term which satisfies the given condition. The nonlinear operator $N(u)$ can be decomposed into an infinite series of polynomial given by

$$N(u) = \sum_{n=0}^{\infty} A_n \qquad (8)$$

where A_n, are the so-called Adomian's polynomials which are evaluated using the formula [6]:

$$A_n(t) = \left(\frac{1}{n!}\right)\left(\frac{d^n}{d\lambda^n}\right) N\left(\sum_{i=1}^{n}(\lambda^i u_i)\right)_{\lambda=0}, \quad n = 0,1,2, \ldots.. \qquad (9)$$

Substituting equations (6) and (8) into (5) gives

$$\sum_{n=1}^{\infty} u_n(t) = F(t) - L^{-1}(Ru) - L^{-1}\left(\sum_{n=0}^{\infty} A_n\right) \qquad (10)$$

Then equating the terms in the linear system of equation (10) gives the recurrent relation

$$\left.\begin{aligned} u_0(x) &= F(t) \\ u_{n+1}(x) &= -L^{-1}(Ru) - L^{-1}(A_n), \quad n \geq 0 \end{aligned}\right\} \qquad (11)$$

However, in practice all the terms of series (10) cannot be determined, and the solution is approximated by the truncated series $\sum_{n=0}^{N} u_n(t)$.

3. **Illustrative examples**

3.1 Example 1

Consider the linear IVP [6]:

$$u'' = (1 + t^2)u, \quad u(0) = 1, \qquad u'(0) = 0, \qquad t \, \epsilon \, [0,1]$$

with the exact solution, u_e where

$$u_e = e^{\frac{1}{2}t^2}$$

Starting with an initial approximation,

$$u_0 = 1$$

and noting that

$$u_{n+1} = \int_0^t \int_0^t (1 + t^2) u_n dt dt$$

we have at $n = 0$

$$u_1 = \int_0^t \int_0^t (1 + t^2) u_0 dt dt = \int_0^t \int_0^t (1 + t^2) dt dt$$

$$= \frac{1}{2}t^2 + \frac{1}{12}t^4$$

at $n = 1$

$$u_2 = \int_0^t \int_0^t (1 + t^2) u_1 dt dt$$

$$= \int_0^t \int_0^t (1 + t^2) \left(\frac{1}{2}t^2 + \frac{1}{12}t^4 \right) dt dt$$

$$= \frac{1}{24}t^4 + \frac{7}{360}t^6 + \frac{1}{672}t^8$$

at $n = 2$

$$u_3 = \int_0^t \int_0^t (1 + t^2) u_2 dt dt$$

$$= \int_0^t \int_0^t (1 + t^2) \left(\frac{1}{24}t^4 + \frac{7}{360}t^6 + \frac{1}{672}t^8 \right) dt dt$$

$$= \frac{1}{720}t^6 + \frac{11}{10080}t^8 + \frac{211}{907200}t^{10} + \frac{1}{88704}t^{12}$$

at $n = 3$

$$u_4 = \int_0^t \int_0^t (1 + t^2) u_3 dtdt$$

$$= \int_0^t \int_0^t (1 + t^2) \left(\frac{1}{720} t^6 + \frac{11}{10080} t^8 + \frac{211}{907200} t^{10} + \frac{1}{88704} t^{12} \right) dtdt$$

$$= \frac{1}{40320} t^8 + \frac{1}{36288} t^{10} + \frac{1201}{119750400} t^{12} + \frac{4867}{3632428800} t^{14} + \frac{1}{21288960} t^{16}$$

$$u(t) = \sum_{n=0}^{N} u_n (t)$$

where $N = 4$

$$\Rightarrow \quad u(t) = 1 + \frac{1}{2} t^2 + \frac{1}{8} t^4 + \frac{1}{48} t^6 + \frac{11}{384} t^8 + \frac{59}{226800} t^{10} + \frac{2551}{119750400} t^{12} + \frac{4867}{3632428800} t^{14} + \frac{1}{21288960} t^{16}$$

Table 1: Comparison of ADM with the exact method

t	u_e (exact solution)	$u(t)$ (ADM)	$\lvert\text{Error}\rvert$
0	1	1	0
0.1	1.005012521	1.005012521	0
0.2	1.020201340	1.020201406	6.6 x10^{-8}
0.3	1.046027860	1.046029569	3.7257499 x10^{-2}
0.4	1.083287068	1.083304133	1.7065 x10^{-5}
0.5	1.133148453	1.133250178	1.01725 x10^{-4}
0.6	1.197217363	1.197654760	1.79473004 x10^{-1}
0.7	1. 277621313	1. 279122548	1.501235 x10^{-3}
0.8	1.377127764	1.381496762	4.368998 x10^{-3}
0.9	1.499302500	1.510512318	1.1209818 x10^{-2}
1	1.648721271	1.674761997	2.6040726 x10^{-2}

3.2 Example 2

Consider the linear IVP [6]:

$$u'' = -u, \qquad u(0) = 1, u'(0) = 0$$

with exact solution $u_e = cost$

Starting with an initial approximation,

$$u_0 = 1$$

and noting that

$$u_{n+1} = \int_0^t \int_0^t -u_n dtdt$$

we have at $n = 0$

$$u_1 = \int_0^t \int_0^t -u_0 dtdt = \int_0^t \int_0^t (-1)\, dtdt$$

$$= -\frac{1}{2}t^2$$

at $n = 1$

$$u_2 = \int_0^t \int_0^t -u_1 dtdt = -\int_0^t \int_0^t -\left(\frac{1}{2}\right) dtdt$$

$$= \frac{1}{24}t^4$$

at $n = 2$

$$u_3 = \int_0^t \int_0^t -u_2 dtdt = -\int_0^t \int_0^t \frac{1}{24}t^4\, dtdt$$

$$= -\frac{1}{720}t^6$$

at $n = 3$

$$u_4 = \int_0^t \int_0^t -u_3 dtdt = \int_0^t \int_0^t \frac{1}{720}t^6\, dtdt$$

$$= \frac{1}{40320}t^8$$

at $n = 4$

$$u_5 = \int_0^t \int_0^t -u_4 dtdt = -\int_0^t \frac{1}{40320}t^8\, dtdt$$

$$= -\frac{1}{3628800}t^{10}$$

$$u(t) = \sum_{n=0}^{N} u_n\,(t)$$

where $N = 5$

$$\Rightarrow \qquad u(t) = 1 - \frac{1}{2}t^2 + \frac{1}{24}t^4 - \frac{1}{720}t^6 + \frac{1}{40320}t^8 - \frac{1}{3628800}t^{10}$$

Table 2: Comparison of ADM with the exact method

t	u_e (exact solution)	$u(t)$ (ADM)	$\lvert\text{Error}\rvert$
0	1	1	0
0.1	0.9950041653	0.9950041653	0
0.2	0.9800665778	0.9800665779	1×10^{-10}
0.3	0.9553364891	0.9553364891	0
0.4	0.9210609940	0.9210609941	1×10^{-10}
0.5	0.8775825619	0.8775825619	0
0.6	0.8253356149	0.8253356149	0
0.7	0.7648421873	0.7648421873	0
0.8	0.6967067093	0.6967067092	1×10^{-10}
0.9	0.6216099683	0.6216099677	6×10^{-10}
1	0.5403023059	0.5403023038	2.1×10^{-9}

3.3 Example 3

Consider the nonlinear IVP [6]:

$$u' = t^2 + u^2, \qquad u(0) = 0$$

Integrating the source term which satisfies the given condition we have the initial approximation

$$u_0 = \int_0^t t^2 = \frac{1}{3}t^3$$

and noting also that

$$u_{n+1} = \int_0^t A_n$$

where

$$A_0 = u_0^2$$

$$A_1 = 2u_0u_1$$

$$A_2 = u_1^2 + 2u_0u_2$$

$$A_3 = 2(u_1u_2 + u_0u_3)$$

$$A_4 = u_2^2 + 2(u_0u_4 + u_1u_3)$$

we have at $n = 0$

$$u_1 = \int_0^t u_0^2\, dt = \int_0^t \frac{1}{9}t^6\, dt = \frac{1}{63}t^7$$

at $n = 1$

$$u_2 = 2\int_t^t u_0\, u_1 dt = 2\int_t^t \left(\frac{1}{3}t^3\right)\left(\frac{1}{63}t^7\right) dt$$

$$= \frac{2}{2079}t^{11}$$

at $n = 2$

$$u_3 = \int_0^t \left(u_1^2 + 2(u_0)(u_2)\right) dt$$

$$= \int_0^t \left(\left(\frac{1}{63}t^7\right)^2 + 2\left(\frac{1}{3}t^3\right)\left(\frac{2}{2079}t^{11}\right)\right) dt$$

$$= \frac{13}{218295}t^{15}$$

at $n = 3$

$$u_4 = \int_0^t (u_1u_2 + 2u_0u_3)\, dt$$

$$= \int_0^t \left(\left(\frac{1}{63}t^7\right)\left(\frac{2}{2079}t^{11}\right) + 2\left(\frac{1}{3}t^3\right)\left(\frac{13}{218295}t^{15}\right)\right) dt$$

$$= \frac{8}{1382535}t^{19}$$

$$u(t) = \sum_{n=0}^{N} u_n\,(t)$$

where $N = 4$

$$\Rightarrow \qquad u(t) = \frac{1}{3}t^3 + \frac{1}{63}t^7 + \frac{2}{2079}t^{11} + \frac{13}{218295}t^{15} + \frac{8}{1382535}t^{19}$$

which coincides with the first five terms of the Taylor series method.

3.4 Example 4

Consider the nonlinear IVP [6]:

$$u' = u + u^2, \qquad u(0) = 1$$

$$Lu = u + u^2 \qquad (12)$$

where $L = \frac{d}{dx}$ and $L^{-1} = \int_0^t dt$

Operating L^{-1} on (12) and imposing the initial condition gives

$u(t) = u_0 + L^{-1}(u + u^2)$

$\Rightarrow$

$$\sum_{n=0}^{\infty} u_n(t) = u_0 + L^{-1}\sum_{n=0}^{\infty} A_n$$

Noting that the initial approximation

$$u_0 = 1$$

and that

$$u_{n+1} = \int_0^t A_n\,, \qquad n \geq 0$$

where

$$A_0 = u_0 + u_0^2$$

$$A_1 = u_1 + 2u_0u_1$$

$$A_2 = u_2 + u_1^2 + 2u_0u_2$$

$$A_3 = u_3 + 2(u_1u_2 + u_0u_3)$$

$$A_4 = u_4 + u_2^2 + 2(u_0u_4 + u_1u_3)$$

at $n = 0$

$$u_1 = \int_0^t A_0 = \int_0^t (u_0 + u_0^2)dt = \int_0^t 2\,dt = 2t$$

at $n = 1$

$$u_2 = \int_0^t A_1 = \int_0^t (u_1 + 2u_0u_1)\,dt = \int_0^t (6t)\,dt = 3t^2$$

at $n = 2$

$$u_3 = \int_0^t A_2 = \int_0^t (u_2 + u_1^2 + 2u_0u_2)\,dt = \int_0^t (13t^2)\,dt = \frac{13}{3}t^3$$

at $n = 3$

$$u_4 = \int_0^t A_3 = \int_0^t \big(u_3 + 2(u_1u_2 + u_0u_3)\big)\,dt = \int_0^t \left(\frac{62}{3}t^3\right)dt = \frac{31}{6}t^4$$

at $n = 4$

$$u_5 = \int_0^t A_4 = \int_0^t (u_4 + u_2^2 + 2(u_0u_4 + u_1u_3))\,dt = \int_0^t \left(\frac{251}{6}t^4\right)dt = \frac{251}{30}t^5$$

$$u(t) = \sum_{n=0}^{N} u_n\,(t)$$

where $N = 5$

$$\Rightarrow \qquad u(t) = 1 + 2t + 3t^2 + \frac{13}{3}t^3 + \frac{31}{6}t^4 + \frac{251}{30}t^5$$

which coincides with the first six terms of the Taylor series method.

4. **Conclusion**

In this paper, the Adomian Decomposition Method for approximating linear and nonlinear Initial Value Problems is implemented, the numerical solutions of ADM is also compared with the exact solution in the first and second examples while in the third and fourth examples they are compared with the Taylor series method.

For the linear IVPs, as seen in Examples 1and 2, the numerical results obtained by using ADM show very good agreement with the exact solutions, and for the nonlinear IVPs, as seen in Examples 3 and 4 the results show that better accuracy can be obtained by accommodating more terms in the decomposition series whereas the Taylor series method suffered from computational difficulties. This makes the ADM efficient, simpler and faster than the classical method of Taylor series solution of IVPs. It also converges to the exact solution.

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