# Application of Variational Iteration Decomposition Method For Solving Linear Differential Equation of Tenth-Order with Boundary conditions.

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**Abstract**

The boundary value problems of higher order have been investigated because of their mathematical importance and the potential for applications in hydro-dynamic and hydro-magnetic stability. Solution to these BVP is of utmost importance in the aforementioned field of study.

In this paper, we make use of a new analytical technique, the variational iteration decomposition method (VIDM), for solving a tenth-order boundary value problem. The anticipated method is a well-designed combination of variational iteration method and decomposition method. The analytical results of the equations have been obtained in terms of convergent series with easily computable components.

Numerical example is given to check the efficiency of this method. The method can be used as an alternative for solving linear boundary value problems.

**Keywords:** Variational iteration decomposition method, linear problems, higher order boundary value problems, decomposition method.

## 1. INTRODUCTION

The main objective of this paper is to apply the variational iteration decomposition method to the solution a tenth-order ordinary differential equation of the type

$$y^{(10)}(x) + P(x)y(x) = Q(x) \qquad\qquad a \le x \le b$$

with boundary conditions

$$y(a) = \alpha_1, \quad y^{ii}(a) = \alpha_2, \quad y^{iii}(a) = \alpha_3, \quad y^{iv}(a) = \alpha_4, \quad y^{v}(a) = \alpha_5$$

$$y(b) = \beta_1, \quad y^{ii}(b) = \beta_2, \quad y^{iii}(b) = \beta_3, \quad y^{iv}(b) = \beta_4, \quad y^{v}(b) = \beta_5,$$

where $\alpha_i, \beta_i \in \mathbb{R}$ $\qquad for \quad i = 1, 2, \cdots, 5$ $\qquad$ and $\quad f \in [a, b]$

It is rare to find research in this area of numerical analysis, but lately, research is strongly in progress. The boundary value problems of higher order have been investigated because of their mathematical importance and the potential for applications in hydro-dynamic and hydro-magnetic stability. A problem of this kind is formed as a result of instability that arises from heating an infinite horizontal layer of fluid from below and in the same way is subjected to rotation. Now when this instability is an ordinary convection, it results into sixth order but if over stability occurs, eight order BVP occurs. If it happens that a uniform magnetic field is also applied across the field in the same direction as that of gravity, and thus is unstable, the ordinary convection set is modeled by the tenth order boundary value problem. Here over stability may occur, thus, it is modeled by an higher order BVP with appropriate boundary condition. Chandrasekhar [1]

Several works had been done on the solution of different higher order B.V.P. Boutayeb and E. H. Twizell [2], developed numerical method for solving sixth order B.V.P. Agarwal [3], also developed Theorem for the existence and uniqueness of solution of higher order

B.V.P e.g. the eight-order while, Siddiqi and Twizell [4], did proffer solution to 10th order B.V.P using Spline method, just to mention a few.

We use the VIDM to solve equivalent system of integral equations efficiently

The VIDM solves effectively, easily and accurately a large class of linear, differential equations with approximate solutions which converge very rapidly to accurate

solutions. An example is given to illustrate the reliability and performance of this method.

## 2. VARIATIONAL ITERATION METHOD

Consider

$$Mu + Nu = g(x) \quad (1)$$

where M and N are linear and non linear operators respectively and

$$g(x) = forcing\ term$$

Constructing a functional iteration we obtain

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda\big(Mu_n(s) + N\bar{u}_n(s) - g(s)\big)d(s) \quad (2)$$

where $\lambda$ Lagrange multiplier [5], which can be obtain optimally through the VIM

The subscript $n$ denotes the $nth$ approximation. $\bar{u}_n$ is a restricted variation such that a slight change $\delta\bar{u}_n = 0$. Thus the equation (2) is called correct functional. Suppose the Lagrange multiplier $\lambda$ can be accurately identified the iteration in (2) can be determined in a single iteration.

In this paper we use the value of $\lambda = -1$

Now, given the system of equation

$$x^{'}(t) = f(x) + g(t) \quad (3)$$

with the initial condition

$$x(0) = \xi \tag{4}$$

we have the equivalent integral equation

$$x(t) = \xi + \int_0^t \big(f(x) + g(s)\big)ds \tag{5}$$

thus for the equation

$$x_i(t) = f_i(x_i) + g_i(t) \qquad i \in \mathbb{N} \tag{6}$$

subject to the boundary condition $x_i(0) = c_i$ we obtain the equivalent integral equation

$$x_1{}^{(k+1)}(t) = x_1^{(t)}(t) - \int_0^t \Big[x_1^{(k)}(T), f_1\Big(\tilde{x}^{(k)}(T), \tilde{x}_2^{(k)}(T), \cdots, \tilde{x}_n^{(k)}(T)\Big) - g_1(T)\Big]\, dT$$

$$x_1{}^{(k+1)}(t) = x_2^{(t)}(t) - \int_0^t \Big[x_2^{(k)}(T), f_2\Big(\tilde{x}^{(k)}(T), \tilde{x}_2^{(k)}(T), \cdots, \tilde{x}_n^{(k)}(T)\Big) - g_2(T)\Big]\, dT$$

⋮

⋮

$$x_n^{(k+1)}(t) = x_n^{(t)}(t) - \int_0^t \Big[x_n^{(k)}(T), f_n\Big(\tilde{x}^{(k)}(T), \tilde{x}_2^{(k)}(T), \cdots, \tilde{x}_n^{(k)}(T)\Big) - g_n(T)\Big]\, dT$$

with $\lambda = -1$

If we begin with the initial approximations $x_i(0) = c_i$, $i \in \mathbb{N}$, then the approximation can be determined completely; finally we approximate the solution

$x_i(t) = \lim_{k\to\infty} x_i^{(n)}(t)$ in the nth term $x_i^{(n)}(t)$ for $i \in \mathbb{N}$

## 3. VARIATIONAL ITERATION DECOMPOSITION METHOD (VIDM)

To illustrate this method we consider the general differential equations of the form

$$Mu + Nu = g(x)$$

where $M$ is a linear operator, $N$ a nonlinear operator and $g(x)$ is the forcing term. According to this method we can construct a correct functional as above stated i.e.

$$U_{n+1}(x) = U_n(x) + \int_0^x \lambda\left(MU_n(x) + N\widetilde{U}_n(s) - g(s)\right) ds \qquad (7)$$

where $\lambda$ is a Langrange multiplier, optimally determined to be $-1$

The subscript $n$ implies nth approximation. $\widetilde{U}_n$ is considered as a restricted variation meaning that $\delta\widetilde{U}_n = 0$. We define the solution of the series as $U(x) = \sum_{i=0}^{\infty} U_{(x)}^{(i)}$ and nonlinear term

$$N(u) = \sum_{i=0}^{\infty} A_n(U_0, \cdots, U_i) \qquad (8)$$

where $A_n$ are the so-called Adomian's polynomials and can be generated for all types of nonlinear terms.

where $$A_n = \left[\left(\frac{1}{n!}\right)\left(\frac{d^n}{d\lambda^n}\right) F(u(\lambda))\right]_{\lambda=0} \qquad (9)$$

Note that, these polynomials $A_n$ called the Adomian's polynomial is used in generating iterative schemes for finding the approximate solutions for a nonlinear higher order differential equation using the variational iteration decomposition method (VIDM).

In this work we shall be considering the tenth order linear differential equation with appropriate boundary condition as we shall illustrate in the example that follows.

## 4. Numerical Applications

In this section, we shall rewrite the tenth-order boundary value problem is its equivalent system of integral equations by using a suitable transformation. The variational iteration decomposition method (VIDM) is applied to solve the resultant system of integral equations.

**Example.** Consider the tenth-order linear boundary value problem

$$y^{(10)}(x) = 10e^x + y(x) \qquad 0 < x < 1$$

with boundary conditions

$$y(0) = 0 \quad y^{ii}(0) = 2 \quad y^{iii}(0) = 4 \quad y^{vi}(0) = 6 \quad y^{v}(0) = 8$$

$$y(1) = e \quad y^{ii}(1) = 3e \quad y^{iv}(1) = 5e \quad y^{vi}(1) = 7e \quad y^{viii}(1) = 7e$$

by applying the transformation we obtain

$$\frac{dy}{dx} = a(x) \qquad \frac{da}{dx} = b(x) \qquad \frac{db}{dx} = e(x) \qquad \frac{de}{dx} = f(x) \qquad \frac{df}{dx} = g(x)$$

$$\frac{dg}{dx} = h(x) \qquad \frac{dh}{dx} = z(x) \qquad \frac{dz}{dx} = p(x) \qquad \frac{dp}{dx} = q(x) \qquad \frac{dq}{dx} = 10e^x + y(x)$$

So that we may have

$$y(0) = 0 \qquad a(0) = A$$

$$b(0) = 2 \qquad e(0) = B$$

$$f(0) = 4 \qquad g(0) = C$$

$$h(0) = 6 \qquad z(0) = D$$

$$p(0) = 8 \qquad q(0) = E$$

From the iteration,

$$y_{k+1} = \int_0^x a_k(s)ds \qquad f_{k+1} = 4 + \int_0^x g_k(s)ds$$

$$a_{k+1} = A + \int_0^x b_k(s)ds \qquad g_{k+1} = C + \int_0^x h_k(s)ds$$

$$b_{k+1} = 2 + \int_0^x e_k(s)ds \qquad h_{k+1} = 6 + \int_0^x z_k(s)ds$$

$$e_{k+1} = B + \int_0^x f_k(s)ds \qquad z_{k+1} = D + \int_0^x p_k(s)ds$$

$$p_{k+1} = 8 + \int_0^x q_k(s)ds \qquad q_{k+1} = E + \int_0^x (10e^s + y_k(s))ds$$

**Initial Point**

$y_0 = 0 \qquad a_0 = A$

$b_0 = 2 \qquad e_0 = B$

$f_0 = 4 \qquad g_0 = C$

$h_0 = 6 \qquad z_0 = D$

$p_0 = 8 \qquad q_0 = E$

**1st Iteration**

$y_1 = Ax \qquad a_1 = A + 2x$

$b_1 = 2 + Bx \qquad e_1 = B + 4x$

$f_1 = 4 + Cx \qquad g_1 = C + 6x$

$h_1 = 6 + Dx \qquad z_1 = D + 8x$

$p_1 = 8 + Ex \qquad q_1 = E + 10e^x$

**2nd Iteration**

$$y_2 = Ax + \frac{2x^2}{2} = Ax + x^2 \qquad x_1 = A + 2x + \frac{Bx^2}{2}$$

$$b_2 = 2 + Bx + \frac{4x^2}{2} = 2 + Bx + 2x^2 \qquad e_2 = B + 4x + \frac{Cx^2}{2}$$

$$f_2 = 4 + Cx + \frac{6x^2}{2} \qquad g_2 = C + 6x + \frac{Dx^2}{2}$$

$$h_2 = 6 + Dx + \frac{8x^2}{2} \qquad z_2 = D + 8x + \frac{Ex^2}{2}$$

$$p_2 = 8 + Ex + 10e^x \qquad q_2 = E + 10e^x + \frac{Ax^2}{2}$$

**3rd Iteration**

$$g_3 = Ax + x^2 + \frac{Bx^3}{6} \qquad a_3 = A + 2x + \frac{Bx^2}{2} + \frac{2x}{3}$$

$$b_3 = 2 + Bx + 2x^2 + \frac{Cx^3}{6} \qquad e_3 = B + 4x + \frac{Cx^2}{2} + x^3$$

$$f_3 = 4 + Cx + 3x^2 + \frac{Dx^3}{6} \qquad g_3 = C + 6x + \frac{Dx^2}{2} + \frac{4x^3}{3}$$

$$h_3 = 6 + Dx + 4x^2 + \frac{Ex^3}{6} \qquad z_3 = D + 8x + \frac{Ex^2}{2} + \frac{5x^3}{3}$$

$$p_3 = 8 + Ex + 10e^x + \frac{Ax^3}{6} \qquad q_3 = E + 10e^x + \frac{Ax^2}{2} + \frac{x^3}{3}$$

**4th Iteration**

$$y_4 = Ax + x^2 + \frac{3x^2}{6} + \frac{x^4}{6} \qquad a_4 = A + 2x + \frac{Bx^2}{2} + \frac{2x^2}{3} + \frac{Cx^4}{24}$$

$$b_4 = 2 + Bx + 2x^2 + \frac{Cx^3}{6} + \frac{x^4}{6} \qquad e_4 = B + 4x + \frac{Cx^2}{2} + x^3 + \frac{Dx^4}{24}$$

$$f_4 = 4 + Cx + 3x^2 + \frac{Dx^2}{6} + \frac{x^4}{3} \qquad g_4 = C + 6x + \frac{Dx^2}{2} + \frac{4x^3}{3} + \frac{Ex^4}{24}$$

$$h_4 = 6 + Dx + 4x^2 + \frac{Ex^3}{6} + \frac{x^4}{3} \qquad z_4 = D + 8x + \frac{Ex^2}{2} + \frac{5x^2}{3} + \frac{Ax^4}{24}$$

$$p_4 = 8 + Ex + 10e^3 + \frac{Ax^3}{6} + \frac{x^4}{12} \qquad q_4 = E + 10e^x + \frac{Ax^2}{2} + \frac{x^3}{3} + \frac{3x^4}{24}$$

**5th Iteration**

$$y_5 = Ax + x^2 + \frac{Bx^2}{6} + \frac{x^4}{6} + \frac{Cx^5}{120} \qquad a_5 = A + 2x + \frac{Bx^2}{6} + \frac{2x^2}{3} + \frac{Cx^4}{24} + \frac{3x^5}{20}$$

$$b_5 = 2 + Bx + 2x^2 + \frac{Cx^3}{6} + \frac{x^4}{4} + \frac{Dx^5}{120} \qquad e_5 = B + 4x + \frac{Cx^2}{2} + x^3 + \frac{Dx^4}{24} + \frac{x^5}{15}$$

$$f_5 = 4 + Cx + 3x^2 + \frac{Dx^3}{6} + \frac{x^4}{3} + \frac{Ex^5}{120} \qquad g_5 = C + 6x + \frac{Dx^2}{2} + \frac{4x^3}{3} + \frac{Ex^4}{24} + \frac{5x^5}{60}$$

$$h_5 = 6 + Dx + 4x^2 + \frac{Ex^3}{6} + \frac{5x^4}{12} + \frac{Ax^5}{120} \qquad z_5 = D + 8x + \frac{Ex^2}{2} + \frac{5x^3}{3} + \frac{Ax^4}{24} + \frac{x^5}{60}$$

$$p_5 = 8 + Ex + 10e^x + \frac{Ax^3}{6} + \frac{x^4}{12} + \frac{3x^5}{120} \qquad q_5 = E + 10e^x + \frac{Ax^2}{2} + \frac{x^3}{3} + \frac{3x^4}{24} + \frac{x^5}{30}$$

Using Taylor's series expansion at the point zero the series solution is this

$$y(x) = \frac{x^{\circ} y^{(\circ)}}{0!}(0) + \frac{x^1 y^{(i)}}{1!}(0) + \frac{x^2 y^{(ii)}}{2!}(0) + \frac{x^3 y^{(iii)}}{3!}(0) + \frac{x^4 y^{(iv)}}{4!}(0) + \cdots + \frac{x^8 y^{(viii)}}{8!}(0) + \cdots +$$

by substitution we have that

$$y^{i}(0) = A, \quad y^{iii}(0) = B, \quad y^{v}(0) = C, \quad y^{vii}(0) = D, \quad y^{ix}(0) = E$$

Hence

$$y(x) = 0 + Ax + \frac{2x^2}{2!} + \frac{Bx^3}{3!} + \frac{4x^4}{4!} + \frac{Cx^5}{5!} + \frac{6x^6}{6!} + \frac{Dx^7}{7!} + \frac{8x^8}{8!} + \frac{Ex^9}{9!} + \frac{10x^{10}}{10!} + \left(\frac{10 + A}{11!}\right)x^{11} + \frac{12x^{12}}{12!} + \left(\frac{10 + B}{13!}\right)x^{13} + \frac{14x^{14}}{14!} + O(x^{15})$$

From this series, we find the value of *A, B, C, D* and E by apply the boundary condition at $x = 1$ thus we solve for

$$y^{(i)}(1), \quad y^{(iii)}(1), \quad y^{(iv)}(1), \quad y^{(vi)}(1) \quad and \quad y^{(viii)}(1)$$

Thus we obtain the system of equations

$$y(1) = \frac{7318002277}{6227020800} + A + \frac{(10+A)}{39916800} + \frac{B}{6} - \frac{(10+B)}{6227020800} - \frac{C}{120} + \frac{D}{504} + \frac{E}{362880} = e$$

$$y^{(ii)}(1) = \frac{1020599719}{239500800} + \frac{(10+A)}{362800} + B + \frac{(10+B)}{39916800} + \frac{C}{6} + \frac{D}{120} + \frac{E}{5040} = 5e$$

$$y^{(iv)}(1) = \frac{13331347}{1814400} + \frac{(10+A)}{5040} + \frac{(10+B)}{362880} + C + \frac{D}{6} + \frac{E}{120} = 5e$$

$$y^{(vi)}(1) = \frac{30049}{2880} + \frac{(10+A)}{120} + \frac{(10+B)}{5040} + D + \frac{E}{6} = 7e$$

$$y^{(viii)}(1) = \frac{4867}{360} + \frac{(10+A)}{6} + \frac{(10+B)}{120} + E = 9e$$

$$\frac{39916801A}{39916800} + \frac{1037836801B}{6227020800} + \frac{C}{120} + \frac{D}{5040} + \frac{E}{362880} = e - \frac{157}{622702080} \tag{10}$$

$$\frac{1}{362880}A + \frac{39916801}{39916800}B + \frac{1}{6}C + \frac{1}{120}D + \frac{1}{5040}E = 3e - \frac{1020606379}{239500800} \tag{11}$$

$$\frac{1}{5040}A + \frac{1}{362880}B + C + \frac{1}{6}D + \frac{1}{120}E = 5e - \frac{4444999}{604800} \tag{12}$$

$$\frac{1}{120}A + \frac{1}{5040}B + D + \frac{1}{6}E = 7e - \frac{212063}{20160} \tag{13}$$

$$\frac{1}{6}A + \frac{1}{120}B + E = 9e - \frac{5497}{360} \tag{14}$$

$$\begin{bmatrix} \frac{39916801}{39916800} & \frac{1037836801}{6227020800} & \frac{1}{120} & \frac{1}{5040} & \frac{1}{362880} \\ \frac{1}{362880} & \frac{39916801}{39916800} & \frac{1}{6} & \frac{1}{120} & \frac{1}{5040} \\ \frac{1}{5040} & \frac{1}{362880} & 1 & \frac{1}{6} & \frac{1}{120} \\ \frac{1}{120} & \frac{1}{5040} & 0 & 1 & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{120} & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \\ E \end{bmatrix} = \begin{bmatrix} e - \frac{157}{622702080} \\ 3e - \frac{1020606379}{239500800} \\ 5e - \frac{4444999}{604800} \\ 7e - \frac{212063}{20160} \\ 9e - \frac{5497}{360} \end{bmatrix}$$

With the aid of appropriate mathematical program we obtained the following constants

$A = 1.150;\quad B = 2.066;\quad C = 5.666;\quad D = 7.499\quad E = 14.001$

The series becomes

$$y(x) = 1.15x + x^2 + 0.344333x^3 + \frac{x^4}{3!} + 0.047216x^5 + \frac{x^6}{5!} + 0.001489x^7 + \frac{x^8}{7!} + 0.000038583x^9$$
$$+ \frac{x^{.10}}{9!} + 2.79331 \times 10^{-7}x^{11} + \frac{x^{12}}{11!} + 1.93768 \times 10^{-9}x^{13} + \frac{x^{14}}{13!} + O(x^{15})$$

Which is the theoretical/numerical series solution to the problem above.

## 5. Conclusion

In this work, we have used the variational iteration decomposition method (VIDM) which is mainly due to M.A Noor and S.T. Mohyud-Din [6], S. Abbasbandy [7] and J.H. He [8], for finding the solution of linear boundary value problems for tenth order. It may be concluded that VIDM is very useful and efficient in finding the analytical solutions for such boundary value problems. The method gives more realistic series solutions that converge very rapidly in physical problems. Thus we conclude this technique can be considered as an efficient method for solving linear B.V. problems.

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