On the direct block multistep method for the solution of fourth order ordinary differential equations

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Abstract

We considered a linear multistep method for the direct solution of fourth order initial value problems of ordinary differential equations. The approach of collocation approximation is adopted in the derivation of the scheme and then the scheme is applied as simultaneous integrator to fourth order initial value problems of ordinary differential equations in non-parallel mode. The method possessed the desirable feature of Runge-Kutta method of being self – starting and eliminated the use of predictors. Numerical examples are given to ensure the efficiency of the method.

Key Words: Linear multistep methods (LMMs); P-stability; Zero-stability; Fourth order; Initial Value Problems (IVPs); Ordinary Differential Equations (ODEs); Interval of periodicity; Predictor-corrector (PC).

1. Introduction

In this paper, the higher order initial value problems of ordinary differential equations of the form:

$$y^{(n)} = f(x,\ y,\ y',\ y'' \ldots\ldots\ldots\ldots\ldots\ldots\ldots y^{(n-1)})\ , \qquad (1)$$
$$y(a) = y_0,\ y^{i}(a) = y_i\ ,\ i = 1(1)n-1,\ n \geq 4$$

is considered for step numbers k ≥ 6. Equation (1) can model many physical problems in elasticity and solid mechanics (with the absence of derivatives on rhs). However, only a limited number of analytical methods are available for solving directly without first reducing it a system of first order differential equations, hence, we resort to numerical methods. Conventionally, equation (1) is solved by first reducing it to an equivalent first–order system and then applying the various methods available for solving system of first order initial value problems. There are considerable literatures on the methods of solution to higher order ODEs (see Lambert [1], [2]; Onumanyi *et al.*, [3]; Fatula [4]; Awoyemi and Kayode [5]). These methods have certain limitations; the computer programs associated with methods are often complicated especially when incorporating the subroutines to supply the starting values for the methods resulting in longer computer time and more computational work Jator [6].

Some eminent scholars have proposed several methods for solving second order ODEs directly without first reducing it to an equivalent first-order system. For instance, Hairer and Wanner [7] proposed Nystom type methods and stated order conditions for determining the parameters of the methods. Awoyemi [8] and Kayode [9] proposed and implemented LMMs in a predictor-corrector mode using the Taylor series algorithm to supply the staring values. Particularly, the improved Numerov method developed by Kayode [9] for the direct solution of general second order IVPs given as:

$$y'_{n+2} = 1/h(y_{n+1} - y_n + h/24)(9f_{n+2} + 26f_{n+1} + f_n), h \neq 0 \qquad (2)$$

order 3, $C_5 = 1/45$. The idea of this method is good but it has certain setbacks that can be circumvented: restriction to predictors for y'_{n+2} in the right hand side of (2) and starting values are required. However, the predictors are developed in the same manner as correctors and they are of lower order to be combined with higher order correctors, thereby reduces the overall accuracy. Moreover, this approach is more costly to implement, in the sense that the subroutines are very complicated to write, since they require special techniques for supplying the starting values and for varying the step-size, which lead to longer computer time and more human effort.

Therefore, this article proposes the development of an order six method which is applied as simultaneous intergrators to fourth order initial and boundary value problems of ordinary differential equations. The method is derived through interpolation and collocation in the spirit of Norsett and Lie [10], it is consistent, zero-stable hence converges.

Moreover, Fatokun and Onumanyi [11] derived second and fourth order two-step discrete finite difference methods by collocation for the first approximation and combined them with the Numerov method for a direct application to general second order initial value problem of ODEs. Furthermore, In Olabode and Yusuph [12], a new block method for special third order ordinary differential equations was derived. An accurate scheme by block method was also proposed and applied as simultaneous intergrators to third order ordinary differential equations Olabode [13].

2. The Material and Method

In this section, the interpolation and collocation procedures is adopted to characterize the linear multistep method that is of interest to us, the right number of interpolation points *(u)*and the right number of collocation points *(v)* are carefully selected.

We approximate the exact solution y(x) by seeking the continuous method $\bar{y}$ (x) of the form:

$$\bar{y}(x) = \sum_{j=2}^{u-1} \alpha_j(x) y_{n+j} + h^4 \sum_{j=0}^{v-1} \beta_j(x) f_{n+j} \qquad (3)$$

where $x \in [a,b]$,and the following notations are introduced. The positive integer $k \geq 4$ denotes the step number of the method (3), which is applied directly to provide solution to (1). So we seek a solution on a given grid point:

$\pi_{N}: a = x_0 < x_1 < ...x_n < x_{n+1} < .... < x_{n+k} .... < x_N = b,$ where π_N is the partition of [a, b] and h = $x_{n+1}-x_n$, n = 0, 1, ……, N is a constant step size of the partition of π_N. The number of interpolation points *u* and the number of distinct collocation points *v* as chosen to satisfy $4 \leq u \leq k,$ and $0 < \mathrm{v} \leq \mathrm{k}+1,$ and we construct k-step multistep collocation method of the form (3) by imposing the following conditions:

$$\bar{y}(x_{n+j}) = y_{n+j}, \ j = 2, 3, ..., u-1 \qquad (4)$$

$$\bar{\mathrm{y}}^{iv}(x_{n+j}) = f_{n+j,} \ j = 0, 1, 2......, v-1 \qquad (5)$$

Equation (4) and (5) lead to a system of (*u*+*v*) equations and *(u+v)* unknown coefficients to be determined. In order to solve this system, we require that the linear k-step methods (3) multistep be defined by the assumed polynomial basis functions:

$$\alpha_j(x) = \sum_{j=2}^{u-1} \alpha_{i+1,j} p_i(x); \ j \in \{2, 3........u-1\} \qquad (6)$$

$$\beta_j(x) = \sum_{j=0}^{u-1} \beta_{i+1,j} p_i(x); \ j \in \{0, 1,........v-1\} \qquad (7)$$

where the constants $\alpha_{i+1,j}$ *and* $h^4\beta_{i+1,j}$, $j=\{0,\ 1,.....u+v-1\}$ are the parameters to be determined from the*(u+v)* x *(u+v)* matrix A, given by

$$M=\begin{pmatrix} \alpha_{2,0} & \alpha_{2,1}\ldots\ldots & \alpha_{2,\text{u-1}} & \text{h}^4\beta_{1,0} & \ldots\ldots & \text{h}^4\beta_{1,v-1} \\ \alpha_{3,0} & \alpha_{3,1}\ldots\ldots & \alpha_{3,\text{u-1}} & \text{h}^4\beta_{2,0} & \ldots\ldots & \text{h}^4\beta_{2,v-1} \\ . & . & . & . & . & \\ . & . & . & . & . & \\ \alpha_{u+v,0} & \alpha_{\text{u+v},1}\ldots & \alpha_{\text{u+v,u-1}} & \text{h}^4\beta_{u+v,0} & \ldots & \text{h}^4\beta_{u+v,v-1} \end{pmatrix} \qquad (8)$$

The interpolation/collocation matrix D is also defined as:

$$D=\begin{pmatrix} p_0(x_{n+2}) & \ldots\ldots\ldots & p_{u+v-1}(x_{n+2}) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \ldots\ldots & .. \\ p_0(x_{n+u-1}) & \ldots\ldots\ \ldots & p_{u+v-1}(x_{n+u-1}) \\ p_0^{iv}(x_n) & \ldots\ldots\ldots\ldots & p_{u+v-1}^{iv}(x_n) \\ p_0^{iv}(x_{n+1}) & \ldots\ldots\ldots & p_{u+v-1}^{iv}(x_{n+1}) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & & \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & & \\ p_0^{iv}(x_{n+v-1}) & \ldots\ldots & p_{u+v-1}^{iv}(x_{n+v-1}) \\ & & . \end{pmatrix} \qquad (9)$$

This system of equations are solved by Gaussian elimination method to obtain the values of parameters a_j's , j = 0, 1, .. which is substituted in approximate solution yields, after some manipulation yield the new continuous method. In Jator [6], matrix inversion approach was employed in the determination of unknowing parameters a_j. The new six-step LMM with continuous coefficients obtained is then applied as simultaneous numerical integrators to fourth order ordinary differential equations (1). The method eliminates the use of predictors by providing sufficiently accurate simultaneous difference equations from a single continuous formula and its derivative. Moreover, this method is cheaper to implement, since it is self-starting and therefore the limitations are circumvented. The continuous method is expressed in the form:

$$y_6(x) = \sum_{j=2}^{k-1} a_j(x) y_{n+j} + \sum_{j=0}^{k} \beta_j(x) f_{n+j} \qquad (10)$$

the coefficients $\alpha_j(x)$ *and* $\beta_j(x)$ of (10) are expressed as functions of $t = (x - x_{n+1})/h$,

$$a_2(t) = \frac{1}{6}(-t^3 + 9t^2 - 26t + 24)$$
$$a_3(t) = \frac{1}{6}(3t^3 - 24t^2 + 57t - 36)$$
$$a_4(t) = \frac{1}{6}(-3t^3 + 21t^2 - 42t + 24)$$
$$a_5(t) = \frac{1}{6}(t^3 - 6t^2 + 11t - 6)$$

$$\beta_0(t) = \frac{h^4}{3628800}(t^{10} - 25t^9 + 255t^8 - 1350t^7 + 3836t^6 - 5040t^5 + 6875t^3 - 4812t^2 - 940t + 1200)$$

$$\beta_1(t) = \frac{h^4}{1814400}(-3t^{10} + 70t^9 - 630t^8 + 2520t^7 - 2058t^6 - 19404t^5 + 75600t^4 - 115960t^3 + 75411t^2 - 9426t - 6120)$$

$$\beta_2(t) = \frac{h^4}{725760}(3t^{10} - 65t^9 + 513t^8 - 1494t^7 - 1428t^6 + 15120t^5 - 148141t^3 + 370200t^2 - 363300t + 128592)$$

$$\beta_3(t) = \frac{h^4}{181440}(-t^{10} + 20t^9 - 138t^8 + 288t^7 + 658t^6 - 2520t^5 - 20478t^3 + 132753t^2 - 228830t + 118248)$$

$$\beta_4(t) = \frac{h^4}{725760}(3t^{10} - 55t^9 + 333t^8 - 522t^7 - 1596t^6 + 5040t^5 - 29963t^3 + 131148t^2 - 232980t + 128592)$$

$$\beta_5(t) = \frac{h^4}{1814400}(-3t^{10} + 50t^9 - 540t^8 + 360t^7 + 1302t^6 - 3780t^5 + 7540t^3 - 8949t^2 + 9870t - 6120)$$

$$\beta_6(t) = \frac{h^4}{3628800}(t^{10} - 15t^9 + 75t^8 - 90t^7 - 364t^6 + 1008t^5 - 1955t^3 + 2088t^2 - 1948t + 1200) \qquad (11)$$

The discrete scheme of (11) is:

$$y_{n+6}-4y_{n+5}+6y_{n+4}-4y_{n+3}+y_{n+2}=\frac{h^4}{15120}(-16f_{n+6}+2574f_{n+5}+10029f_{n+4}$$
$$+2504f_{n+3}+54f_{n+2}-30f_{n+1}+5f_n).$$
$$\textit{with}\ \text{order P}=8\ \text{i.e C}_{10}=-3.373015873\ell^{-04}. \qquad (12)$$

$f_n = f(x_n, y_n)$, $f_{n+1} = f(x_{n+1}, y_{n+1})$, $f_{n+2} = f(x_{n+2}, y_{n+2})$, $f_{n+3} = f(x_{n+3}, y_{n+3})$,

$f_{n+4} = f(x_{n+4}, y_{n+4})$, $f_{n+5} = f(x_{n+5}, y_{n+5})$, $f_{n+6} = f(x_{n+6}, y_{n+6})$

The first, second and third derivatives of (11) are found, noting that

$$t=(x-x_{n+1})/h \qquad (13)$$

Evaluating equation (11) at $x = x_n$, $x = x_{n+1}$ and $x = x_{n+6}$, with the additional equations obtained from the first, second and third derivative functions yield the following integrators:

$$y_{n+6}-4y_{n+5}+6y_{n+4}-4y_{n+3}+y_{n+2}=\frac{h^4}{15120}(-16f_{n+6}+2574f_{n+5}+10029f_{n+4}$$
$$+2504f_{n+3}+54f_{n+2}-30f_{n+1}+5f_n)$$

$$y_n+4y_{n+5}-15y_{n+4}+20y_{n+3}-10y_{n+2}=\frac{h^4}{15120}(25f_{n+6}-234f_{n+5}+10770f_{n+4}$$
$$+41920f_{n+3})+20745f_{n+2}+2370f_{n+1}+4f_n)$$

$$y_{n+1}+y_{n+5}-4y_{n+4}+6y_{n+3}-4y_{n+2}=\frac{h^4}{15120}(5f_{n+6}-51f_{n+5}+2679f_{n+4}$$
$$+9854f_{n+3}+2679f_{n+2}-51f_{n+1}+5f_n)$$

$$hZ_0-\frac{13}{3}y_{n+5}+\frac{31}{2}y_{n+4}+114y_{n+3}+47y_{n+2}=\frac{h^4}{907200}(-1535f_{n+6}$$
$$+14418f_{n+5}-696540f_{n+4}-2845040f_{n+3}-1848915f_{n+2}-435930f_{n+1}-7658f_n)$$

$$h^2Z_0'+3y_{n+5}-10y_{n+4}+11y_{n+3}-4y_{n+2}=\frac{h^4}{302400}(-502f_{n+6}+2902f_{n+5}$$
$$+140915f_{n+4}-7286000f_{n+3}+575180f_{n+2}-322282f_{n+1}+19823f_n)$$

$$h^3Z_0''-y_{n+5}+3y_{n+4}-3y_{n+3}+y_n=\frac{h^4}{120960}(1335f_{n+6}-9608f_{n+5}$$
$$+10459f_{n+4}-156920f_{n+3}-55219f_{n+2}-176608f_{n+1}-36799) \qquad (14)$$

Comment 2.1

It is worth to note that the method (12) is the main method that can be implemented in the PC mode in that case predictors and starting values will be required.

3. Analysis and the Implementation of the method

In this section, we analyze the method for consistency, error constant, zero stability and convergence. According to Fatunla [4] and Lambert [1], the linear difference operator L is defined as:

$$L[y(x);h]=\sum_{j=0}^{k}[\alpha_j y(x+jh)-h^4\beta_j y^{iv}(x+jh)] \tag{15}$$

where y(x) is the exact solution to (1) and is assumed to be sufficiently differentiable. We now invoke the Taylor‘s theorem to obtain

$$L[y(x);h]=C_0 y(x)+C_1 hy'(x)+\ldots+C_q h^q y^{(q)}(x)+o(h^{q+2}) \tag{16}$$

whose coefficients C_q, q = 0,1,.. are constants independent of y(x) and given as:

$$C_0=\sum_{j=0}^{k}\alpha_j$$

$$C_1=\sum_{j=1}^{k}j\alpha_j$$

$$C_q=\frac{1}{q!}\left[\sum_{j=1}^{k}j^q\alpha_j-q(q-1)\sum_{j=1}^{k}j^{q-2}\beta_j\right] \tag{17}$$

The order p of the difference operator L[y(x); h] is a unique integer p such that

$C_q = 0$, q = 0(1) p+1, $C_{P+2} \neq 0$ (Henrici [14]).

In order to analyze the method for zero stability, equations (15) to (17) are written as block method given by the matrix difference equation. The first-block of LMM for the fourth order initial value problem (ivp) designated by equation (1) can be expressed by the following matrix difference equation:

$$A^{(o)}.\, y_q = A^{(1)}.y_{q-1} + h^4B^{(0)}.F_q + BF_{q-1} \tag{18}$$

where, $y_q = (y_{n+1}, y_{n+2},\ldots, y_{n+6})^T, y_{q-1} = (y_{n-5}, y_{n-4},\ldots, y_n)^T$

$f_{q-1} = (f_{n-5}, f_{n-4},\ldots, f_n)^T$, $f_q = (f_{n+1}, f_{n+2},\ldots, f_{n+6.})^{T,}$ q = 0, 1, ...

and n= 0, 6….and the matrix A^o is an identity matrix of dimension 6.

$$\begin{bmatrix} 0 & 1 & -4 & 6 & -4 & 1 \\ 0 & -10 & 20 & -15 & 4 & 0 \\ 1 & -4 & 6 & -4 & 1 & 0 \\ 0 & 47 & 114 & 31/2 & -13/3 & 0 \\ 0 & -4 & 11 & -10 & 3 & 0 \\ 0 & 0 & -3 & 3 & -1 & 0 \end{bmatrix} \begin{pmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+3} \\ y_{n+4} \\ y_{n+5} \\ y_{n+6} \end{pmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} \begin{pmatrix} y_{n-5} \\ y_{n-4} \\ y_{n-3} \\ y_{n-2} \\ y_{n-1} \\ y_n \end{pmatrix}.$$

$$+h^4 \left[\begin{bmatrix} \frac{-1}{504} & \frac{1}{280} & \frac{313}{1890} & \frac{3343}{5040} & \frac{429}{2520} & \frac{1}{945} \\ \frac{237}{1512} & \frac{4149}{3024} & \frac{524}{189} & \frac{1077}{1512} & \frac{-13}{840} & \frac{5}{3024} \\ \frac{-17}{5040} & \frac{893}{5040} & \frac{4927}{5040} & \frac{893}{5040} & \frac{-17}{5040} & \frac{1}{3024} \\ \frac{-307}{181440} & \frac{89}{5600} & \frac{-11609}{15120} & \frac{-35563}{11340} & \frac{-41087}{20160} & \frac{-14531}{30240} \\ \frac{-161141}{151200} & \frac{28759}{15120} & \frac{-18215}{756} & \frac{28183}{60480} & \frac{1451}{151200} & \frac{-251}{151200} \\ \frac{-5519}{3780} & \frac{-55219}{120960} & \frac{-3923}{3024} & \frac{10459}{120960} & \frac{-1201}{15120} & \frac{89}{8064} \end{bmatrix} \cdot \begin{pmatrix} f_{n+1} \\ f_{n+2} \\ f_{n+3} \\ f_{n+4} \\ f_{n+5} \\ f_{n+6} \end{pmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \frac{1}{3024} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{3780} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{3024} \\ 0 & 0 & 0 & 0 & 0 & \frac{-3829}{453600} \\ 0 & 0 & 0 & 0 & 0 & \frac{19823}{302400} \\ 0 & 0 & 0 & 0 & 0 & \frac{-36799}{120960} \end{bmatrix} \cdot \begin{pmatrix} f_{n-5} \\ f_{n-4} \\ f_{n-3} \\ f_{n-2} \\ f_{n-1} \\ f_n \end{pmatrix} \right]. \quad (19)$$

For easy analysis, equation (19) is normalised by multiplying its matrix A^0 and A^1 with the inverse of A^0 i.e $(A^0)^{-1}$. The first characteristic polynomial of the improved block method is

$$\rho(R) = \det[RA^0 - A^1]$$

$$= \det \left[R \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \right]$$

$$\rho(R) = \begin{vmatrix} R & 0 & 0 & 0 & 0 & -1 \\ 0 & R & 0 & 0 & 0 & -1 \\ 0 & 0 & R & 0 & 0 & -1 \\ 0 & 0 & 0 & R & 0 & -1 \\ 0 & 0 & 0 & 0 & R & -1 \\ 0 & 0 & 0 & 0 & 0 & R-1 \end{vmatrix} = 0 \qquad (20)$$

$$\rho(R) = R^5(R-1) = 0 \Rightarrow \ R_1 = R_2 = R_3 = R_4 = R_5 = 0, \ \ R_6 = 1$$

which implies zero-stability. The block method is also consistent, as it has the order p>1. Hence, the convergence of the method is asserted Henrici [14].

4. Numerical Experiments and Results

This section deals with the numerical experiments and results using the algorithm proposed for fourth order initial value problems.

1. $$y^{iv} = \frac{Q(x)}{EI}; \qquad 0 \le x \le 1$$

$$y(0) = 0, \ y'(0) = \frac{\phi(x)}{48}, \ y''(0) = 0, \ y'''(0) = \frac{-3\phi(x)}{8}$$

$$\phi(x) = E = I = 1$$

Theoretical solution

$$y(x) = \frac{\phi(x)}{48}(2x^4 - 3x^3 + x)$$

TABLE 1: Accuracy comparison of the direct solution of problem (1), with the result obtained when the same problem (1) is reduced to the first order system, h= 0.1

X	Exact solution y(x)	LMM of order 6 y-computed	Errors of the direct solution of problem (1)	Errors (when reduced to first order system)
0.1	0.002025000	0.002020835	4.1650000E-07	1.6171E-06
0.2	0.003733333	0.003666721	6.6612333E-06	5.4933E-06
0.3	0.004900000	0.004562906	3.3709400E-05	1.0129E-05
0.4|	0.0054000000	0.00433499	1.0650100E-05	1.4023E-05
0.5	0.005208333	0.002609012	2.5993213E-05	1.5677E-05
0.6	0.004400000	-0.000988581	5.3885810E-05	1.3590E-05
0.7	0.003150000	-0.006831112	9.9811120E-04	6.2619E-06
0.8	0.001733333	-0.015292039	1.7025372E-04	7.8069E-06
0.9	0.0005250000	-0.02674555	2.7270550E-03	3.0116E-05

Table 1 shows that, the maximum absolute error of the new linear multistep method for the direct solution of the problem (1) is 4.1650000E-07 while the maximum absolute error when the same problem (1) is reduced to system of first order is 1.6171000E-06 which implies that they compare favorably.

2. $$y^{(4)} = -9y + x$$
$$y(0) = 0,\ y'(1) = 0,\ y''(0) = 0,\ y'''(1) = 0$$

TABLE 2: The approximate solution of problem (2), with h = 0.1

X	y-computed	LMM of order 6 y-computed
0.1	0.0832783	8.327830E-02
0.2	0.0164637	1.646370E-02
0.3	0.0242184	2.421840E-02
0.4|	0.0314076	3.140760E-02
0.5	0.0378557	3.785570E-02
0.6	0.433984	4.339840E-01
0.7	0.047888	4.788800E-02
0.8	0.0511976	51.19760E-02
0.9	0.0532275	5.322750E-02
1	0.0539119	5.391190E-02

5. Conclusion

The new block linear multistep method was proposed for the direct solution of fourth order initial value problems of ordinary differential equations eliminated the use of predictor and it is also more accurate and faster than the conventional (step-step) integration procedures. Furthermore, the direct method is more advantageous than the traditional reduction method to the first order system in the sense that it is cost effective and attractive from computational point of view.

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