A PERTURBED RUNGE-KUTTA METHOD FOR SOLUTION OF SECOND ORDER DIFFERENTIAL EQUATIONS

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Abstract

A fourth order Runge-kutta method with a perturbation term $\in_{n+1}(x)$ is derived for solving directly general and stiff second order ordinary Differential Equations of the form $y'' = f(x, y, y')$, $y(x_0) = y_0$, $y'(x_0) = y'_0$. This is a reconstruction and upgrade of the Runge-kutta methods for first order equations. We define a linear transformation T on set of ordered-three tuples. This enables us to generate four function evaluations that are repeatedly and directly used for the solution y_{n+1}. The scheme can be used to solve explicit, implicit, stiff, oscillatory or periodic, non-linear second order ODEs very efficiently. Numerical examples are used to compare with other recent methods.

Keywords:- Direct, perturbed, linear transformation, explicit, implicit, stiff, non-linear ODEs.

1.0 Introduction

Many researchers have developed different methods for solving higher order ODEs. Some of these methods are Computational methods in ODEs [1], Block method for second order ODEs [2], Algorithm for general second order ODEs [3], Multi-step collocation methods [4], etc. In most of these methods, reductions of higher order into system of first order (Indirect methods) were adopted to solve them. These techniques are good but have many auxiliary equations with complicated and long Computer Programs, leading to waste of time, energy, etc.

Some existing direct methods are Runge-Kutta-Nystrom [5], improved continous method for direct solution of second order ODEs [6], Numerov method [7], Improved block method for special Second order ODEs [8], Efficient numerical method for highly oscillatory ODEs [9]. These schemes are also good but most of them can only solve explicit or special second order ODEs of the form y"=f(x); y"=f(y) and y"=f(x,y) only. Their degree of accuracy are low, thus there is need to develop algorithms for solving both special and general second order ODEs of the form $y'' = f(x, y, y')$, with higher degree of accuracy.

2.0 Methodology

In this section, we shall derive the perturbed Runge-Kutta method of the form

$$y(x) = y^h(x) + \epsilon(x) \quad \text{--} \quad (2.1)$$

where $y^h(x)$ is the direct fourth order Runge-Kutta solution with step size h and $\epsilon(x)$ is a small perturbation term on $y^h(x)$ which will increase the degree of our accuracy. To do this, we consider the general second order differential equations (ODEs)

$$a\,\frac{d^2y}{dx^2} + \,b\frac{dy}{dx} + cy = g(x,y), \quad y(x_0) = y_0,\ y'\,(x_0) = y'_0 \quad \text{---------------} \ (2.2)$$

$were$ a, b, c are real constants or scalar functions of x and y.

We rewrite (2.2) in the form:

$$y'' = \frac{d^2y}{dx^2} = f(x,y,y'), \quad y(x_0) = y_0,\ y'\,(x_0) = y'_0 \quad \text{------------------} \ (2.3)$$

Let T be a transformation on the set of ordered three tuples

$(x,y,y') \overset{T}{\rightarrow} \mathcal{R}$, defined as follows

$T(x,y,y') = y',\ T'(x,y,y') = f(x,y,y')$.

This Transformation T is well-defined because if there exist Z such that

$T(x,y,y') = T(x,z,z')$ and $T'(x,y,y') = T'(x,z,z')$

Then by definition of T

$y' = z'$ and $f(x,y,y') = f(x,z,z')$, thus this implies that $y = z$, ie T is well defined.

Let $h(0 < h < 1)$ be a scalar. We define

$$k_1 = hT(x,y,y') = hy',\ k'_1 = hT'(x,y,y') = hf(x,y,y') = m_1$$

$$k_2 = hT\left(x + \tfrac{1}{2}h, y + \tfrac{1}{2}k_1, (y + \tfrac{1}{2}k_1)'\right) = h\left(y' + \tfrac{1}{2}k'_1\right) = h(y' + \tfrac{1}{2}m_1)$$

$$k'_2 = hT'\left(x + \tfrac{1}{2}h, y + \tfrac{1}{2}k_1, y' + \tfrac{1}{2}k'_1\right) = hf\left(x + \tfrac{1}{2}h, y + \tfrac{1}{2}k_1, y' + \tfrac{1}{2}k'_1\right) =$$

$$hf\left(x + \tfrac{1}{2}h, y + \tfrac{1}{2}hy', y' + \tfrac{1}{2}m_1\right) = m_2$$

$$k_3 = hT\left(x + \tfrac{1}{2}h, y + \tfrac{1}{2}k_2, y' + \tfrac{1}{2}k'_2\right) = h\left(y' + \tfrac{1}{2}k'_2\right) = h(y' + \tfrac{1}{2}m_2)$$

$$k'_3 = hT'\left(x + \tfrac{1}{2}h, y + \tfrac{1}{2}k_2, y' + \tfrac{1}{2}k'_2\right) = hf\left(x + \tfrac{1}{2}h, y + \tfrac{1}{2}k_2, y' + \tfrac{1}{2}k'_2\right) =$$
$$hf\left(x + \tfrac{1}{2}h, y + \tfrac{1}{2}h\left(y' + \tfrac{1}{2}m_1\right), y' + \tfrac{1}{2}m_2\right) = m_3$$

$$k_4 = hT(x + h, y + k_3, y' + k'_3) = h(y' + k'_3) = h(y' + m_3)$$

$$k'_4 = hT'(x + h, y + k_3, y' + k'_3) = hf(x + h, y + k_3, y' + k'_3)$$

$$= hf(x + h, y + h\left(y' + \tfrac{1}{2}m_2\right), y' + m_3) = m_4$$

The solution with step size h is:

$$y^h(x) = {}^{h}_{y}(x + h) = y(x) + \frac{1}{6}[(k_1 + k_4) + 2(k_2 + k_3)]$$

$$= y(x) + \tfrac{1}{6}[h(2y' + m_3) + 2h\left(2y' + \tfrac{1}{2}m_1 + \tfrac{1}{2}m_2\right)]$$

$$= y(x) + \tfrac{1}{6}[2hy' + hm_3 + 4hy' + m_1 + m_2]$$

$$y^h(x + h) = y_1 = y + hy' + \tfrac{1}{6}h(m_1 + m_2 + m_3) \text{ ------------------------- (2.4)}$$

$$y'^h(x + h) = y' + \tfrac{1}{6}[(k'_1 + k'_4) + 2(k'_2 + k'_3)]$$

$$= y'(x) + \frac{1}{6}[(m_1 + m_4) + 2(m_2 + m_3)]$$

Now taking two- half steps of h, we define

$$m^*_1 = \frac{1}{2}hf(x, y), \qquad m^*_2 = \frac{1}{2}hf(x + \frac{1}{4}h, y + \frac{1}{4}hy', y' + \frac{1}{2}m^*_1)$$

$$m^*_3 = \frac{1}{2}hf(x + \frac{1}{4}h, y + \frac{1}{4}h(y' + \frac{1}{2}m^*_1), y' + \frac{1}{2}m^*_2)$$

$$m^*{}_4 = \frac{1}{2}hf(x + \frac{1}{2}h, y + \frac{1}{2}h(y' + \frac{1}{2}m^*{}_2), y' + m^*{}_3)$$

$$y_{1/2} = y + \frac{1}{2}\ hy' + \frac{1}{12}h(m^*{}_1 + m^*{}_2 + m^*{}_3)$$

$$y'{}_{1/2} = y' + \frac{1}{6}\ [(m^*{}_1 + m^*{}_4) + 2(m^*{}_2 + m^*{}_3)]$$

$$m^*{}_5 = \frac{1}{2}hf(x + \frac{1}{2}h, y_{\frac{1}{2}}, y'{}_{\frac{1}{2}})$$

$$m^*{}_6 = \frac{1}{2}hf(x + \frac{3}{4}h, y_{\frac{1}{2}} + \frac{1}{4}h\, y'{}_{\frac{1}{2}}\ , y'{}_{\frac{1}{2}} + \frac{1}{2}\ m^*{}_5\ )$$

$$m^*{}_7 = \frac{1}{2}hf(x + \frac{3}{4}h, y_{\frac{1}{2}} + \frac{1}{4}h\ (y'{}_{\frac{1}{2}}\ + \frac{1}{2}m^*{}_5), y'{}_{\frac{1}{2}} + \frac{1}{2}\ m^*{}_6\ )$$

$$m^*{}_8 = \frac{1}{2}hf(x + h, y + \frac{1}{2}h\ (y'{}_{\frac{1}{2}}\ + \frac{1}{2}m^*{}_6), y'{}_{\frac{1}{2}} +\ m^*{}_7\ )$$

$$y^{\frac{h}{2}}(x + h) = y_1^{\frac{h}{2}} = y_{\frac{1}{2}} + \frac{1}{2}hy'{}_{\frac{1}{2}} + \frac{1}{12}h[m^*{}_5 + m^*{}_6 + m^*{}_7] \text{ ------------------ (2.5)}$$

$$y_1^{',\frac{h}{2}} = y_{\frac{1}{2}}^{',\frac{h}{2}} + \frac{1}{6}[(m^*{}_5 + m^*{}_8) + 2(m^*{}_6 + m^*{}_7)]$$

Putting $x = x_n$, $y = y_n$ then equations (2.4) and (2.5) becomes

$$y_{n+1}^{h} = y_n + hy'{}_n + \frac{1}{6}h(m_1 + m_2 + m_3) \text{ ------------------------------------ (2.6)}$$

$$y_{n+1}^{\prime h} = y'{}_{n+1} + \frac{1}{6}[(m_1 + m_4) + 2(m_2 + m_3)]$$

$$y_{n+1}^{\frac{h}{2}} = y_{n+\frac{1}{2}} + \frac{1}{2}hy'_{n+\frac{1}{2}} + \frac{1}{12}h(m^*_5 + m^*_6 + m^*_7) \text{ ----------------------------- } (2.7)$$

$$y'^{\frac{h}{2}}_{n+1} = y'_{n+\frac{1}{2}} + \frac{1}{6}[(m^*_5 + m^*_8) + 2(m^*_6 + m^*_7)]$$

y_{n+1}^{h} *and* $y_{n+1}^{\frac{h}{2}}$ are the approximate solutions of (2.3) with step size h and two-halve steps of h respectively. Every Runge-Kutta method of order k can be expanded into Taylor's series and as $k \to \infty$ both y_{n+1}^{h} *and* $y_{n+1}^{\frac{h}{2}}$ converges to the same exact solution. Now letting $k \to \infty$, we have

$$y_{n+1} = y_{n+1}^{h} + Ch^{k+1} \text{ --- } (2.8)$$

$$y_{n+1} = y_{n+1}^{\frac{h}{2}} + C(\frac{h}{2})^{k+1} \text{ --- } (2.9)$$

where Ch^{k+1} is the first neglected infinite term of the series when $y(x_n + h)$ is expanded into Taylor's series.

Subtracting (2.9) from (2.8), we have

$$0 = y_{n+1}^{h} - y_{n+1}^{\frac{h}{2}} + [\, Ch^{k+1} - C(\frac{h}{2})^{k+1}]$$

$$Ch^{k+1} = \frac{2^{k+1}}{2^{k+1} - 1}(y_{n+1}^{\frac{h}{2}} - y_{n+1}^{h})$$

We take the least upper bound of $Ch^{k+1} = \epsilon_{n+1}$ as perturbation term in (2.8)

Since $\frac{2^{k+1}}{2^{k+1}-1} \leq \frac{2^{k+1}+1}{2^{k+1}-1}$ for all k_1.

$$Ch^{k+1} = \frac{2^{k+1}}{2^{k+1}-1}\left(y_{n+1}^{\frac{h}{2}} - y_{n+1}^{h}\right) \leq \frac{2^{k+1}+1}{2^{k+1}-1}\,(y_{n+1}^{\frac{h}{2}} - y_{n+1}^{h}) = \epsilon_{n+1}$$

Since it is a Runge-Kutta method of order $k = 4$, our perturbed term ϵ_{n+1} is given as:

$$\epsilon_{n+1} = \frac{33}{31}(y_{n+1}^{\frac{h}{2}} - y_{n+1}^{h}) \quad \text{--- (2.10)}$$

$$\epsilon'_{n+1} = \frac{33}{31}(y_{n+1}^{\prime\frac{h}{2}} - y_{n+1}^{\prime h})$$

Thus the proposed perturbed method for solving (2.3) is

$$y_{n+1} = y_{n+1}^{(h)} + \epsilon_{n+1} \quad \text{-- (2.11)}$$

$$y'_{n+1} = y_{n+1}^{\prime h} + \epsilon'_{n+1}$$

where $y_{n+1}^{(h)}$, $y_{n+1}^{(\frac{h}{2})}$, ϵ_{n+1} are already defined by (2.6), (2.7) and (2.8) respectively.

3.0 Numerical Experiments.

In this section we illustrate the performance of the scheme (2.11) using some examples of initial value problems used by some eminent Scholars whose exact solutions are known on the interval $0 \leq x \leq 0.5$. The errors arising from the computed and exact values at the grid points are compared in the Tables below:

Example 1:

$$y'' = -y \ , y(0) = 1, y'(0) = 1, h = 0.1$$

Exact solution is $y(x) = cosx + sinx$

Table 1a: Comparison of theoretical and approximate solutions of example 1,

$0 \leq x \leq 0.5, h = 0.1$

Grid points	Exact solution	Numerov method [7]	Hybrid block method [10]	Perturbed RK method (This study)
0.1	1.09483758191	1.094837379	1.094837761	1.09483758199
0.2	1.17873590863	1.178735501	1.178736267	1.17873590864
0.3	1.25085669578	1.250856127	1.250857196	1.25085669543
0.4	1.3104793363	1.310478608	1.310479976	1.31047933567
0.5	1.35700810049	1.357007268	1.357008835	1.35700809965

Table 1b: Errors of Numerical solutions of example 1

Grid points	Numerov method [7]	Hybrid block method [10]	Perturbed RK method (This study)
0.1	2.03 E (-7)	1.79 E (-7)	8.0 E (-11)
0.2	4.8 E (-7)	3.58 E (-7)	1.0 E (-11)
0.3	5.69 E (-7)	5.0 E (-7)	3.5 E (-10)
0.4	7.28 E (-7)	6.4 E (-7)	6.3 E (-10)
0.5	8.33 E (-7)	7.34 E (-7)	8.4 E (-10)

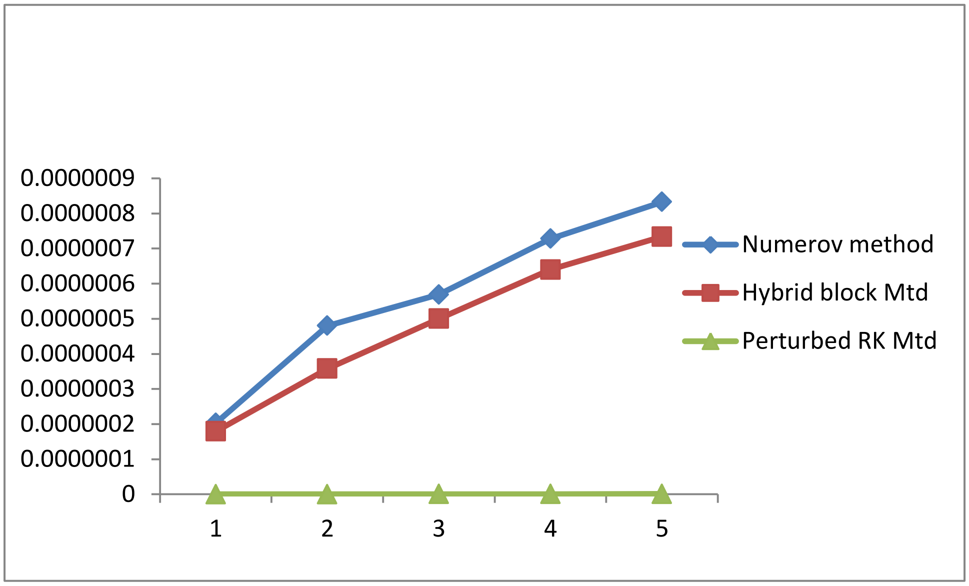

Figure 1: Error Graph of Numerical Solutions of example 1

Example 2:

$y'' - 3y' + 2y = x\ ,$

$y(0) = 1, y'(0) = 0, h = 0.1$

Exact solution: $y(x) = e^{x} - \frac{3}{4}\,e^{2x} + \frac{1}{2}x + \frac{3}{4}$

Table 2: Comparison of theoretical and approximate solutions of example 2 with Errors

Grid points	Exact solution	Perturbed RK method (This study)	Error
0.1	0.98911885000	0.98911886539	1.54 E(-8)
0.2	0.95253423506	0.95253425767	2.26 E(-8)
0.3	0.88326970800	0.88326974102	3.30 E(-8)
0.4	0.77266900200	0.77266905192	4.99 E(-8)
0.5	0.61000990000	0.61000986423	3.58 E(-8)

Example 3:

$y'' = x(y')^2$

$y(0) = 1, y'(0) = 0.5, h = 0.1$

Exact solution: $y(x) = 1 + \frac{1}{2} In(\frac{2+x}{2-x})$

Table 3: Comparison of theoretical and approximate solutions of example 3

Grid points	Exact solution	Perturbed RK method (This study)	Error
0.1	1.05004173927	1.05004172924	3.0 E(-11)
0.2	1.10033534773	1.10033534765	8. 0 E(-11)
0.3	1.15114043593	1.15114043581	1.2 E(-10)
0.4	1.20273255405	1.20273255387	1.8 E(-10)
0.5	1.25541281188	1.25541281162	2.6 E(-10)

4.0 Conclusion

The proposed method is direct and simpler compared with indirect method of reducing second order ODEs to system of first order simultaneous equations which involves many equations per iteration of y_{n+1}. It maximises the order of RK method for second order ODEs. The method can both be used for explicit and implicit ODEs. The results in example 1 are more accurate and stable than Numerov and Hybrid methods (See error graphs). Our proposed scheme is a one- step method because we use the data of just one preceding step in contrast to multistep methods which in each step uses data from several preceding steps. This makes our method simpler. The method is Self-starting, no preliminary calculations are required. The equation and its initial conditions are sufficient to provide the next point on the solution curve.

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