Stability of multi-level scheme for the solution of wave equation

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Abstract

We study the stability of a multi-level scheme for the solution of wave equation. The function u, is the wave amplitude which depends on the x and t variables. The function u, as seen in this work is an admissible function, hence, it can be expanded using the Taylor series. A practical result for stability criteria for the multi-level difference scheme for the solution of wave equation is given in a proposition due to Von Neumann. This result shows that the multi-level difference scheme is stable or unconditional stable.

Key words: Stability, multi-level difference scheme, Von Neuman, difference equation, central difference scheme and symmetric matrix.

INTRODUCTION

We consider the wave equation

$$\frac{\partial^2 u(x,t)}{\partial t^2} = \frac{c^2 \partial^2 u(x,t)}{\partial x^2}$$

u(0,t) = u(1,t) = 0 and u(x,0) = 0 (1.1)

where u is wave amplitude and c^2 = 1 is the speed of wave (see Dass [1] and Tejumola [2]).

We consider also a two level scheme

$$\frac{u_j^{n+1} - u_j^{n-1}}{2\Delta t} = \frac{u_{j+1}^n - u_{j-i}^n}{2\Delta x} \qquad (1.2)$$

which is usually called the central difference scheme [3]. The function u is admissible, it takes values in suitable sub-spaces of the space of definition of the problem under consideration [4]. Hence, the Taylor series expansion is applied to it [5] and satisfies equation (1.1) we are suppose to solve. A practical result for stability criteria for multi-level difference scheme for the solution of wave equation is given in a proposition due to Von Neumann.

Proposition (Von Neumann, [6]): If λ(Δt,k) is an eigenvalue of the amplification matrix G (Δt,k) of a difference scheme, then the necessary and sufficient condition for stability are

i. $|\lambda| \leq$ 0 (Δt)

ii. G(Δt,k) is a symmetric matrix

iii. The scheme involves only one depended variable.

However, the Von Neumann condition is necessary and sufficient for all level scheme for the wave equation irrespective of the number of dependent variable involve [7].The condition obtained by this method is more accurate than the Jacobi's iteration method or the Gauss-Seidel method [8].

2 Main Result

We consider the difference scheme

$$\frac{u_j^{n+1}-u_j^{n-1}}{2\Delta t}=\beta\left[\frac{u_{j+1}^{n}-u_j^{n-1}-u_j^{n+1}+u_{j-1}^{n}}{(\Delta x)^2}\right] \tag{2.1}$$

$$u_j^{n+1}-u_j^{n-1}=\frac{2\Delta t\beta}{(\Delta x)^2}\left[u_{j+1}^{n}-u_j^{n+1}-u_j^{n-1}+u_{j-1}^{n}\right] \tag{2.2}$$

put $\alpha=\frac{2\Delta t\beta}{(\Delta x)^2}$

$$u_j^{n+1}-u_j^{n-1}=\alpha\left[u_{j+1}^{n}-u_j^{n+1}-u_j^{n-1}+u_{j-1}^{n}\right] \tag{2.3}$$

Let $u_j^n=\lambda^n e^{ikx}$

$$u_j^{n+1}=\lambda^{n+1}e^{ikx}$$

$$u_{j+1}^{n+1}=\lambda^{n+1}e^{ik(x+\Delta x)}$$

Substituting in (2.3) we have

$$\lambda^{n+1}e^{ikx}-\lambda^{n-1}e^{ikx}=\alpha[\lambda^n e^{ik(x+\Delta x)}-\lambda_{n+1}e^{ikx}-\lambda^{n-1}e^{ikx}+\lambda^n e^{ikx} \tag{2.4}$$

$$\lambda^2-1=\alpha[\lambda e^{ik\Delta x}-\lambda^2-1+\lambda e^{ik\Delta x}]$$

$$\lambda^2-1=-\alpha\lambda^2-\alpha+\alpha\lambda(e^{ik\Delta x}+e^{-ik\Delta x}) \tag{2.5}$$

$$=\alpha\lambda^2-\alpha+2\alpha\lambda\cos k\Delta x$$

Substituting in (2.5) gives

$$(1+\alpha)\lambda^2 - 2\alpha\lambda c - (1-\alpha) = 0 \quad (2.6)$$

where c = coskΔx

$$\lambda^2 - \frac{2\alpha c\lambda}{1+\alpha} - \frac{(1-\alpha)}{(1+\alpha)} = 0 \quad (2.7)$$

The stability condition is

$|\lambda| \leq 1,$

which implies that, for the equation

$x^2 - 2bx + c = 0$

(i) $|c| \leq 1,$

(ii) $|b| \leq 1$

Hence, $\frac{\alpha c}{1+\alpha} = \frac{\alpha}{1+\alpha} < 1, \forall\alpha$ (2.8)

that is, we must have

$$|\lambda_{1,.}\lambda_2| = \left|\frac{1-\alpha}{1+\alpha}\right| \leq 1, \forall\alpha \quad (2.9)$$

also, $|b| = \left|\frac{\alpha c}{1+\alpha}\right| \leq \left|\frac{\alpha}{1+\alpha}\right| \leq 1, \forall\alpha$ (2.10)

The scheme is always stable or unconditional stable.

3. Conclusion

The Von Neumann condition used in this work shows that the multi-level differences scheme is stable.

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