Lax convergence principle of a linear multi-step scheme for solving parabolic equation

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Abstract

A linear multi-step scheme is considered for the solution of a parabolic equation. The function u, takes values in suitable subspaces of the space of definition of the system under study. Such a function u is said to be admissible and so, it therefore admit the Taylor series expansion. A practical result for convergence was obtained on the application of the equivalent theorem of Lax. Based on this theorem, we establish that the scheme is consistent and stable which implied that the scheme converge.

Key word: consistency, stability, convergence difference equation, linear multi-step scheme, equivalent theorem of Lax and wave equation.

Introduction

We consider the heat flow equation

$u_t = u_{xx}$ (1.1)

where t and x are the time and space coordinates respectively in the region

$\Re = \left\{ a \le x \le b \right\} x\ [t \ge 0]$

with appropriate initial and boundary conditions

u(0,t) = u(1,t) = 0 and u(x,0) = 0 (1.1a)

The region $\Re$ is replaced by set a of points $\Re_i$ which are the vertices of a grid of points (m,n) where x = a+mh, t = nk with Mh = b-a, M being integer. The quantities k and h are mesh sizes in the time and space directions respectively (see Jain [1] and Tejumola [2]).

We write the finite difference scheme for equation (1.1) as

$$\frac{u_{i,n+1} - u_{i,n}}{\Delta t} = \frac{u_{i-1,n} - 2u_{i,n} + u_{i+1,n}}{(\Delta x)^2} \qquad (1.2)$$

where n is the subscript of space at time level [3]. Equation (1.2) is in the explicit form and satisfies the Bendre Schmidt recurrence relation [4]. The function u takes values in suitable subspaces of the space of definition of the system under study. Such a function is said to be admissible [5]. Because of the admissibility of the function u, it therefore admit the Taylor series expansion [6].

We now write equation (1.2) as

$$u_{i,n+1} - u_{in} = \frac{\Delta t}{(\Delta x)^2}\left[u_{i+1,n} - 2u_{i,n} + u_{i-1,n}\right] \tag{1.3}$$

Let $\frac{\Delta t}{(\Delta x)^2} = \lambda$ (1.4)

$$u_{i,n+1} - u_{i,n} = \lambda u_{i+1,n} - 2\lambda u_{i,n} + \lambda u_{i-1,n} \tag{1.5}$$

$$u_{i,n+1} = \lambda u_{i+1,n} + (1-2\lambda)\, u_{i,n} + \lambda u_{i-1,n} \tag{1.6}$$

where λ lies in the range $0 < \lambda < 1$, where λ is a numerical parameter [7]

A practical result for convergence of the linear multi-step scheme for the solution of the parabolic equation is given in the equivalent theorem of Lax.

Equivalent Theorem of Lax [8]: For a properly well possed initial value problem and a finite difference equation to it that satisfies the consistency condition the stability is the necessary and sufficient condition for convergence

We therefore study the following properties one by one.

2 Main Results

2.1 Consistency

We consider the finite difference scheme

$$\frac{u_i^{\,n+1} - u_i^{\,n}}{\Delta t} = \frac{u_{i+1}^n - 2u_i^{\,n} + u_{i-1}^n}{(\Delta x)^2} \tag{2.1}$$

for the solution of the parabolic equation

$u_t = u_{xx}$

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \frac{1}{\Delta t}\,[u(x,t) + \Delta t\, u_t\,(x,t) + \frac{(\Delta t)^2}{2} + u_{tt}\,(x,t)\ldots - u(x,t)] \qquad (2.2)$$

$$= u_t + \frac{\Delta t}{2} u_{tt} \ldots\ldots + \qquad (2.3)$$

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} - u_t = \frac{\Delta t}{2} u_{tt} + \ldots\ldots = o\,(\Delta t) \longrightarrow 0 \qquad (2.4)$$

$$u_{i+1} = u(x,t) + \Delta x u_x^n(x,t) + \frac{(\Delta x)^2}{2} u_{xx}^n(x,t) + \frac{(\Delta x)^3}{3!} u_{xxx}(x,t) + \ldots \qquad (2.5)$$

$$u_{i-1}^n = u(x,t) - \Delta x u_x^n(x,t) + \frac{(\Delta x)}{2!} u_{xx}(x,t) - \frac{(\Delta x)^3}{3!} u_{xxx}^n(x,t) + \ldots \qquad (2.6)$$

$$= 0\,(\Delta x)^2$$

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} - \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{(\Delta x^2} - (u_t - u_{xx}) = O(\Delta t) + O(\Delta x)^2 \qquad (2.7)$$

Since the total energy will approach zero as $\Delta t \longrightarrow 0$ and $\Delta x \longrightarrow 0$, we say that the scheme is consistent for the parabolic equation it is supposed to solve.

2.2 Stability

We consider the finite difference scheme

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{(\Delta x)^2} \qquad (2.8)$$

Let $\lambda = \dfrac{\Delta t}{(\Delta x)^2}$

$$u_i^{n+1} - u_i^n = \lambda(u_{i+1}^n - 2u_i^n + u_{i-1}^n)$$

$$u_i^{n+1} = \lambda u_{i+1}^n + (1-2\lambda)u_i^n + \lambda u_{i-1}^n \tag{2.9}$$

Put $u_i^n = \phi(t)e^{i\alpha x}$

$$u_i^{n+1} = \phi(t+\Delta t)e^{i\alpha x}$$

$$\phi(t+\Delta t)e^{i\alpha x} = \lambda\phi(t)e^{i\alpha(x+\Delta x)} + (1-2\lambda)\phi(t)\, e^{i\alpha x} + \lambda\phi(t)\, e^{i\alpha(x-\Delta x)} \tag{2.10}$$

$$= \phi(t)[\lambda e^{i\alpha\Delta x}\, e^{i\alpha x} + (1-2\lambda)e^{i\alpha x} + \lambda e^{i\alpha x}\, e^{-i\alpha x}] \tag{2.11}$$

$$= \phi(t)\, [(1-2\lambda) + \lambda e^{i\alpha\Delta x} + e^{-i\alpha\Delta x})] \tag{2.12}$$

$$= \phi(t)\, [(1-2\lambda) + \lambda\, (\cos\alpha\Delta x + i\sin\alpha\, \Delta x + \cos\alpha\Delta x - i\sin\alpha\, \Delta x] \tag{2.13}$$

$$= \phi(t)\, [(1-2\lambda) + 2\lambda\, (\cos^2 \alpha\, \frac{\Delta x}{2} - \text{Sin}^2\, \alpha\, \frac{\Delta x}{2})]$$

$$= \phi(t)\, [(1-2\lambda) + 2\lambda(1-2\, \text{Sin}^2\, \alpha \frac{\Delta x}{2})]$$

$$= \phi(t)[1-2\lambda+2\lambda-4\lambda \text{Sin}^2\, \alpha\, \frac{\Delta x}{2}]$$

$$= \phi(t)\, [1-4\lambda \text{Sin}^2\, \alpha\, \frac{\Delta x}{2}] \tag{2.14}$$

$$\frac{\phi(t+\Delta t)}{\phi(t)} = 1-4\lambda s^2 \tag{2.15}$$

where $s = \text{Sin}\, \alpha\, \frac{\Delta x}{2}$

For stability we must have

$$\left|\frac{\phi(t+\Delta t)}{\phi(t)}\right| \le 1,$$

that is, $\left|\frac{\phi(t)+\Delta t)}{\phi(t)}\right| = |1-4\lambda s^2| \leq 1$ (2.16)

that is,

$|1 < 1 - 4\lambda s^2 < 1$

that is, $-2 < -4\lambda s^2$ and $-4\lambda s^2 < 0$

that is, $4\lambda s^2 < 2$, $4\lambda\ s^2 > 0$

$\lambda s^2 \leq ½$

Since $/s^2/ = 1$, we have that

$\lambda \leq ½$ and $\lambda > 0$

$= 0 \leq \lambda \leq ½$

Of course consistency plus stability implies convergence

3. Conclusion

The linear multi-step scheme as seen in the work is consistent, stable and convergent.

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