# STABILIZABILITY FOR NONLINEAR SYSTEMS THAT ADMIT SOME LYAPUNOV-LIKE FUNCTIONS.

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**ABSTRACT:**
In this paper, the stabilizability of dynamical systems which admits Lyapunov –like functions was investigated, with example as an illustration.

**KEY WORDS**: Stabilizability, Lyapunov-like functions, Exponential stabilizability.

## INTRODUCTION.

Consider the Nonlinear dynamical systems

$$\dot{x} = f(t, x(t)), \quad t \geq 0 \tag{1.1}$$

The stabilizability of Nonlinear systems (1.1) is a concept that is very important in most dynamical systems on the account of its theoretical interest to Mathematicians as well as Engineers. As observed by Eke [1], stabilizability is derived from stability. Stability of the systems (1.1) can be investigated through linearization method, but in general, the most powerful technique is the method called direct method. For this method one usually assumes the existence of the so called Lyapunov function, a positive definite function with negative derivative along the trajectory of the systems.

Much work has been done in the stabilizability of nonlinear dynamical systems.. Dayawansa and Martin [2] studied asymptotic stabilization of two and three dimensional nonlinear control systems. Phat and Piyapong [3] studied stabilization problems for class of linear nonautonomuos systems with norm bounded controls, using Lyapunov function technique. Prieur [4] studied robust asymptotic stabilization of nonlinear systems. Phat and Kiet [5] studied asymptotic stability of nonlinear time-

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varying differential equations by Lyapunov direct method. Albertini and Sontag [6] verified that for time-varying systems, global asymptotic controllability to a given subset of the state space is equivalent to the existence of a continuous control Lyapunov function with respect to the set. Zabczyk [7],on his own, studied robustness of exponentially stabilizable systems and the relationships between controllability and stabilizability.Mohamed [8] considered feedback stabilization for the infinite dimensional bilinear systems and gave sufficient conditions for exponential and weak stabilization. Andrey *et al* [9] considered the problem of boundary stabilization of a one dimensional wave equation with an internal spatially antidamping term.

In this paper, we investigate the stabilizability of nonlinear dynamical systems when it admits Lyapunov-like functions. The Lyapunov-like functions proposed in this paper are differentiable functions.( i e at least $C'$- functions.)

**NOTATIONS AND DEFINITIONS.**

$C^0$ = the set of continuous functions.

$C^1$ = the set of functions whose first derivative is continuous.

$\Re$ = the one dimensional Euclidean space.

$\Re^+$ = the set of all non-negative real numbers.

$\Re^n$ = the n-finite dimensional Euclidean space.

$C^\infty$ = the class of continuously differentiable functions defined on $\Re^+$

$W = \Re^+ \times D$, *where* $D \subset \Re^m$ is an open set containing the origin.

**DEFINITIONS**

**Definition 1**.
A function $V(t,x):\mathfrak{R}^{+}\times D\rightarrow\mathfrak{R}$ is called Lyapunov-like function for (1.1) if V(t,x) is continuously differentiable in $t\in\mathfrak{R}^{+}$ *and in* $x\in D$, and there exist positive constants $\lambda_1,\lambda_2,\lambda_3,k,p,q,r,\delta$ such that

1. $\lambda_1\|x\|^{p}\leq V(t,x)\leq\lambda_2\|x\|^{q}$, $\forall\,(t,x)\in W$ (1.2)

and

2. $D_f V(t,x)\leq\lambda_3\|x\|^{r}+ke^{-\delta t}$, $\forall t\geq 0, x\in D_{\backslash\{0\}}$. (1.3)

**Definition 2**.
The dynamical system (1.1) is exponentially stabilizable if its zero solution $x(t,x_0)$ satisfies

$$\|x(t,x_0)\|\leq\beta(\|x_0,t_0)\|e^{-\delta(t-t_0)},\ \forall\,t\geq t_0 \qquad (1.4)$$

where $\beta(h,t):\mathfrak{R}^{+}\times\mathfrak{R}^{+}\rightarrow\mathfrak{R}^{+}$ is a non-negative function increasing in $h\in\mathfrak{R}^{+}$ and $\delta$ is a positive constant.

**Definition 3.**
A set of vector functions $(x_1,x_2,...,x_k)$ is said to be linearly independent over a finite interval I if for every set of non-zero vector $(a_1,a_2,...,a_k)$, there exists a subset J of I with positive measures such that

$$\sum_{i=1}^{k}a_i x_i(t)\neq 0,\ \forall\,t\in J.$$

**Definition 4**
A map T(x) is said to be uniformly asymptotically stabilizable if and only if

$$\|T(x)\|\rightarrow 0\ as\ x\rightarrow\infty.$$

**Definition 5.** (Zabczyk) [7]
Systems (1.1) is said to be exponentially stabilizable if there exist a solution x(.) and constants $\alpha, M, and\ \delta > 0$ such that for all $t \geq 0\ and\ all\ x_0, |x_0| < \delta$

$$|x(t, x_0)| \leq Me^{-\alpha t}|x_0|. \tag{1.5}$$

**Definition 6. (Aniaku) [10]**
A map T(x) is uniformly exponentially stabilizable if and only if there exist two positive real numbers $M > 1, w$ such that for any x, we have

$$\|T(x)\| \leq Me^{-wx}. \tag{1.6}$$

## 2 . **PROPOSITIONS**

The propositions which will help us to establish our main theorem are as follows;

**Proposition 2.1** A dynamical system which is exponentially stabilizable is stabilizable.

**Proof**;
If the dynamical systems (1.1) is exponentially stabilizable, then there exist a solution x(.) and some constants $\alpha, M, \delta > 0$ such that for all $t \geq 0$ and all $x_0, |x_0| < \delta,$ then (1.5) applies. i e $|x(t, x_0)| \leq Me^{-\alpha t}|x_0.|$ We note that $Me^{-\alpha t}|x_0| \to 0\ as\ t \to \infty$ .

From Eke [1] stabilizability of dynamical systems is implied by stability and asymptotic stability implies stability. It follows that $|x(t, x_0)|$ is stabilizable.

**Proposition 2.2** Let x (t) be the maximal solution of (1.1). If x(t) is continuous and

$$d^+x(t) \leq f(t, x(t)), \forall\, t \geq t_0 \qquad \text{then}$$

$$x(t) - (x_0) \leq \int_{t_0}^{t} f(s, x(s))ds, \forall\, t \geq t_0. \tag{2.1}$$

## 3. MAIN RESULT.

We now state our main theorem.

**Theorem 3.1** The dynamical systems

$$\dot{x} = f(t, x(t)), \ t \geq 0$$
$$x(t_0) = x_0, \ t_0 \geq 0 \qquad (3.1)$$

is uniformly exponentially stabilizable if it admits some Lyapunov-like functions V(t,x)

and the following two conditions hold for all $(t, x) \in W = \Re^{+} \times D$

1. $$\delta > \frac{\lambda_3}{|\lambda_2|^{\frac{r}{q}}} \qquad (3.2)$$

and

2. There exists $\gamma > 0$ such that

$$V(t,x) - |V(t,x)|^{r/q} \leq \gamma e^{-\delta t} \qquad (3.3)$$

**Proof.**
Consider any initial time $t_0 \geq 0$ and let $x(t, f) = x(t)$ say, be any solution of (3.1) with

$x(t_0) = x_0$. Let us set

$$U(t,x) = V(t,x) e^{M(t-t_0)}, \quad M = \lambda_3 \big/ |\lambda_2|^{r/q}, \text{ then}$$

$$\dot{U}(t,x) = D_f \, |V(t,x)| e^{M(t-t_0)} + MV(t,x) e^{M(t-t_0)}.$$

Assuming the inequality (1.3) for all $t \geq t_0, x \in D,$ we have

$$\dot{U}(t,x) \leq (-\lambda_3 \|x\|^r + ke^{-\delta t}) e^{M(t-t_0)} + MV(t,x) e^{M(t-t_0)}$$

By using the inequality (1.2), we have

$$\|x\|^q \geq \frac{V(t,x)}{\lambda_2} \text{ or equivalently } -\|x\|^r \leq -\left|\frac{V(t,x)}{\lambda_2}\right|^{r/q}.$$

So we have

$$\dot{U}(t,x) \le \left\{ -\left|V(t,x)\right|^{r/q} \frac{\lambda_3}{\left|\lambda_2\right|^{r/q}} + ke^{-\delta t} \right\} e^{M(t-t_0)} + MV(t.x)e^{M(t-t_0)}.$$

Since $\lambda_3 \big/ \left|\lambda_2\right|^{r/q} = M$ for all $t \ge 0,$ we have

$$\dot{U} \le M\left\{ V(t,x) - \left|V(t,x)\right|^{r/q} \right\} e^{M(t-t_0)} + ke^{(M-\delta)(t-t_0)}$$

Making use of (3.3), we have $\dot{U}(t,x) \le (M\gamma + k)e^{(M-\delta)(t-t_0)}.$

By Proposition 2.2, we have on integrating both sides from $t_0$ to t

$$U(t,x) - U(t_0,x_0) \le \int_{t_0}^{t} (M\gamma + k)e^{(M-\delta)(s-t_0)} ds$$

$$= \frac{M\gamma + k}{M-\delta}[e^{(M-\delta)(t-t_0)} - 1]$$

Setting $\delta_1 = -(M-\delta),$ we see from (3.2) that $\delta_1 > 0$ so that

$$U(t,x) \le U(t_0,x_0) + \frac{M\gamma + k}{\delta_1} - \frac{M\gamma + k}{\delta_1} e^{(M-\delta)(t-t_0)} \le U(t_0,x_0) + \frac{M\gamma + k}{\delta_1}.$$

Since $U(t_0,x_0) = V(t_0,x_0) \le \lambda_2 \left\|x_0\right\|^q$, we get $U(t,x) \le \lambda_2 \left\|x_0\right\|^q + \dfrac{M\gamma + k}{\delta_1}$

Setting $\lambda_2 \left\|x_0\right\|^q + \dfrac{M\gamma + k}{\delta_1} = \beta(\left\|x_0\right\|),\ x_0 > 0$ we have

$$U(t,x) \le \beta(\left\|x_0\right\|),\ \forall\, t \ge t_0 \qquad (3.4)$$

On the other hand, from (1.2) it follows that

$$\lambda_1 \left\|x\right\|^p \le V(t,x) \quad \Rightarrow \left\|x\right\| \le \left\{ \frac{V(t,x)}{\lambda_1} \right\}^{1/p} \qquad (3.5)$$

Setting

$V(t,x) = U(t,x)e^{-M(t-t_0)}$ into (3.5) we get

$$\|x\| \le \left\{ \frac{U(t,x)}{\lambda_1} e^{-M(t-t_0)} \right\}^{1/p} \tag{3.6}$$

From (3.4) and (3.6) we have

$$\|x\| \le \left\{ \frac{\beta(\|x_0\|)}{\lambda_1} e^{-M(t-t_0)} \right\}^{1/p}$$

i.e

$$\|x\| \le \left\{ \frac{\beta(\|x_0\|)}{\lambda_1} \right\}^{1/p} e^{-M/p(t-t_0)}, \forall\, t \ge t_0 \tag{3.7}$$

This shows that (3.1) is exponentially stabilizable and so it is stabilizable.

Example.

Consider a nonlinear differential equation

$$\dot{x} = -\frac{1}{2} x^{2/3} + x e^{-3t}, \quad t > 0 \tag{3.8}$$

Taking Lyapunov –like function $V(t,x): \Re^+ \times D \to \Re^+$ *as* $V(t,x) = x^4$ where

$D = \{x : |x| < 1\}$. We note that $|x|^5 \le V(t,x) \le |x|^4, \forall\, x \in D$

Then condition (1.2) is satisfied with $\lambda_1 = \lambda_2 = 1, p = 5,$ *and* $q = 4$

Also

$$\dot{V}(t,x) = 4x^3 \dot{x} = 4x^3 \left\{ -\frac{1}{2} x^{2/3} + x e^{-3t} \right\}$$

$$= -2x^{11/3} + 4x^4 e^{-3t} \le -2x^{11/3} + 4e^{-3t}$$

So,

$$\dot{V}(t,x) \le -2x^{11/3} + 4e^{-3t}$$

We also see that conditions (1.2) and (1.3) are satisfied with

$\lambda_3 = 2, k = 4, \delta = 3$ *and* $r = 11/3$. Moreover conditions (3.2) and (3.3) of Theorem 3 are

also satisfied for $V(t,x) - |V(t,x)|^{r/q} = x^4 - |x^4|^{(11/3 \div 4)} = x^4 - x^{11/3} = x^{11/3}(x^{1/3} - 1) \le 0 \le e^{-3t}$

(3.8) is exponentially stabilizable and so is stabilizable.

**REFERENCES.**

[1] Eke A.N.(2000) Stabilizability For Linear Feedback Observable Systems. Journal of the Nigerian Mathematical Society. Vol 19, pp 59-68.

[2] Dayawansa W.P and Martin C.F (2010).Asymptotic Stabilization of Low Dimensional Systems. Technical Research. University of Maryland, College Park. Havard. Pp1-15.

[3]. Phat Vu. N. and Piyapong N. (2010) Stabilization of Linear Non-autonomous Systems with Norm Bounded Controls, pp1-14 . (http.www.math.ac.vn/library/download/e-print/05/pdf/vnphat0503.pdf)

[4].Prieur. C.(2006) Robust Stabilization of Nonlinear Control Systems By Means of Hybrid Feedbacks. Rend Sem Mat Univ. Pol. Torino. Vol. 64,. No. 1, pp25-38.

[5].Phat Vu. N. and Kiet T.T.(2000) Lyapunov Stability of Nonlinear Time-Varying Differential Equations. ACTA MATHEMATICA VIETNAMICA. Vol. 25, No. 2, pp 231-249.

[6] Albertini F and Sontag E.D (1983) Control-Lyapunov Functions For Time Varying Set Stabilization. Nonlinear Anal. Vol.7, No.11, pp 1163-1173.

[7]. Zabczyk .J (1989). Some Comments on Stabilizability. Applied Mathematics and Optimization. Vol. 19, pp 1-9.

[8]. Mohamed O. (2010) Exponential and Weak Stabilization of Constrained Bilinear Systems. SIAM J. Control Optimization. Vol. 48, No. 6, pp3962-3974.

[9].Andrey S, Eduardo C. and Miroslav K. (2010) Boundary Stabilization of a 1-D Wave Equation with In-Domain Antidamping . SIAM J.Control Optim. Vol. 48, No. 6, pp 4014-4031.

[10]. Aniaku S.E. (2010) Stabilizability of Systems with Perturbations. Ph.D. Thesis Presented to U.S.T. Port-Harcourt, Rivers State Nigeria.