A note on the rhotrix system of equations

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Abstract

Rhotrix system of equations has been dealt with in the literature where one of the equations is treated and a number of solvability conditions were suggested. In this paper we extend this problem to the case when all the systems were considered to be solved simultaneously. Rhotrix is an object that lies in some way between $n \times n$ dimensional matrices and $(2n-1)\times(2n-1)$ dimensional matrices and representation of vectors in rhotrix is different from the representation of vectors in matrix.

Keywords: rhotrix, rhotrix vectors, rhotrix system of equations

AMS Subject Classifications [2010]: 15A06,15A15

1. Introduction

The concept of rhotrix was first introduced by Ajibade [1] as an extension of the initiative on matrix-tertions and matrix-noitrets suggested by Atanassov and Shannon [2]. The initial algebra and analysis of rhotrices was presented in [1]. The multiplication of rhotrices defined by Ajibade [1] is as follows: Let R and Q be two rhotrices such that

$$R = \left\langle \begin{matrix} & a & \\ b & h(R) & d \\ & e & \end{matrix} \right\rangle \text{ and } Q = \left\langle \begin{matrix} & f & \\ g & h(Q) & j \\ & k & \end{matrix} \right\rangle. \tag{1}$$

The addition and multiplication of rhotrices R and Q defined by Ajibade [1] are as follows:

$$R+Q = \left\langle \begin{matrix} & a+f & \\ b+g & h(R)+h(Q) & d+j \\ & e+k & \end{matrix} \right\rangle,$$

$$R \circ Q = \left\langle \begin{matrix} & ah(Q)+fh(R) & \\ bh(Q)+gh(R) & h(R)h(Q) & dh(Q)+jh(R) \\ & eh(Q)+kh(R) & \end{matrix} \right\rangle.$$

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Another multiplication method for rhotrices called *row-column multiplication* was introduced by Sani [3] in an effort to answer some questions raised by Ajibade [1]. The row-column multiplication method is in a similar way as that of multiplication of matrices and is illustrated using the matrices R and Q defined in (1) as follows:

$$R \circ Q = \left\langle \begin{matrix} & af+dg & \\ bf+eg & h(R)h(Q) & aj+dk \\ & bj+ek & \end{matrix} \right\rangle .$$

A generalization of the row-column multiplication method for n-dimensional rhotrices was given by Sani [4]. That is: given n-dimensional rhotrices $R_n = \langle a_{ij}, c_{lk} \rangle$ and $Q_n = \langle b_{ij}, d_{lk} \rangle$ the multiplication of R_n and Q_n is as follows:

$$R_n \circ Q_n = \langle a_{i_1 j_1}, c_{l_1 k_1} \rangle \circ \langle b_{i_2 j_2}, d_{l_2 k_2} \rangle = \left\langle \sum_{i_2 j_1 = 1}^{t} (a_{i_1 j_1} b_{i_2 j_2}), \sum_{l_2 k_1 = 1}^{t-1} (c_{l_1 k_1} d_{l_2 k_2}) \right\rangle, \; t = (n+1)/2 .$$

The method of converting a rhotrix to a special matrix called 'coupled matrix' was suggested by Sani [5]. This idea was used to solve systems of $n \times n$ and $(n-1)\times(n-1)$ matrix problems simultaneously. The concept of vectors, one-sided system of equations and eigenvector eigenvalue problem in rhotrices were introduced by Aminu [6]. A necessary and sufficient condition for the solvability of one sided system of rhotrix was also presented in [6]. If a system is solvable it was shown how a solution can be found. Rhotrix vector spaces and their properties were presented by Aminu [7]. Linear mappings and square root of a rhotrix were discussed by Aminu in [8] and [9] respectively.

To the author's knowledge a problem consisting of all rhotrix system of equations was not treated. It is the aim of this paper to introduce a problem consisting of all rhotrix system of equations and the task is to find a solution to these systems simultaneously.

2. Rhotrix and its basic properties

Let $t = (n+1)/2$ for $n \in \mathbb{N}$. By 'rhotrix' we understand an object that lies in some way between $n \times n$ dimensional matrices and $(2n-1)\times(2n-1)$ dimensional matrices. That is an n-dimensional rhotrix is the following:

$$R_n = \langle a_{ij}, c_{lk} \rangle = \left\langle \begin{array}{ccccccc} & & & a_{11} & & & \\ & & a_{21} & c_{11} & a_{12} & & \\ & a_{31} & c_{21} & a_{22} & c_{12} & a_{13} & \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ a_{t1} & \dots & \dots & \dots & \dots & \dots & a_{1t} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ & a_{tt-2} & c_{t-1t-2} & a_{t-1t-1} & c_{t-2t-1} & a_{t-2\,t} & \\ & & a_{t\,t-1} & c_{t-1\,t-1} & a_{t-1\,t} & & \\ & & & a_{tt} & & & \end{array} \right\rangle, \quad (2)$$

where $a_{ij}, a_{lk} \epsilon \mathbb{R}$ for $i, j = 1, 2, ..., t$ and $k, l = 1, 2..., t-1$. It is straightforward to verify that the addition of *n*-dimensional rhotrices $R_n = \langle a_{ij}, c_{lk} \rangle$ and $Q_n = \langle b_{ij}, d_{lk} \rangle$ is

$$R_n + Q_n = \langle a_{ij}, c_{lk} \rangle + \langle b_{ij}, d_{lk} \rangle = \langle (a_{ij} + b_{ij}), (c_{lk} + d_{lk}) \rangle, \quad (3)$$

where $i, j = 1, 2, ..., t$ and $l, k = 1, 2...t-1$ with $t = (n+1)/2$.

We will use throughout this paper the row-column multiplication method of rhotrices.

Rhotrix vectors (either row vectors or column vectors) can be represented in *t* different ways where $t = (n+1)/2$. This is different compared to vectors in matrices that can be represented in a unique way. For more information on rhotrix vectors the reader is referred to [6] and [7].

There is a unique representation of any t-dimensional matrix vector while any *n*-dimensional rhotrix vector can be represented in *t* different ways where $t = (n+1)/2$. This can be illustrated as follows: A *3*-dimensional matrix column vector is uniquely given as

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

whereas, a *5*-dimensional rhotrix column vector could be any of

$$\left\langle \begin{array}{ccccc} & & x_1 & & \\ & x_2 & 0 & 0 & \\ x_3 & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & \\ & & 0 & & \end{array} \right\rangle, \left\langle \begin{array}{ccccc} & & 0 & & \\ & 0 & 0 & x_1 & \\ 0 & 0 & x_2 & 0 & 0 \\ & x_3 & 0 & 0 & \\ & & 0 & & \end{array} \right\rangle \text{ or } \left\langle \begin{array}{ccccc} & & 0 & & \\ & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & x_1 \\ & 0 & 0 & x_2 & \\ & & x_3 & & \end{array} \right\rangle. \quad (4)$$

We use the notation introduced in [6] as

$$\langle x^{nj} \rangle \quad (5)$$

to represent the main rhotrix column vector and the main rhotrix row vector is denoted by

$$\langle x^{in} \rangle \quad (6)$$

where $i, j = 1, 2, ..., t$ with $t = (n+1)/2$. Thus, the three rhotrix column vectors in (4) are $\langle x^{51} \rangle, \langle x^{52} \rangle$ and $\langle x^{53} \rangle$ respectively. Similarly, we denote the other columns and rows which are not the main as

$$\langle x^{n-1k} \rangle$$

and

$$\langle x^{ln-1} \rangle$$

respectively, where $k, l = 1, 2, ..., t-1$. Consequently, the *n*-dimensional matrix column vectors will be represented as x^{nj}. The *n*-dimensional identity rhotrix will be denoted by I_n and is given by

$$I_n = \left\langle \begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & 1 & 0 & & \\ & 0 & 0 & 1 & 0 & 0 & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ & 0 & 0 & 1 & 0 & 0 & \\ & & 0 & 1 & 0 & & \\ & & & 1 & & & \end{array} \right\rangle.$$

We also denote by 0 the usual zero, which is the neutral element under addition and for convenience we use the same symbol to denote any rhotrix or rhotrix vector whose every component is 0.

We will now summarize some basic properties of rhotrices that will be used later on. The following properties hold for *n*-dimensional rhotrices A, B and C over $\mathbb{R}$ and $\alpha \epsilon \mathbb{R}$:

$$A + 0 = 0 + A = A$$
$$A + B = B + A$$
$$(A+B)+C = A+(B+C)$$
$$\alpha(A+B) = \alpha A + \alpha B$$
$$A(B+C) = AB + AC$$
$$A(BC) = (AB)C$$
$$AI_n = A = I_n A$$

3. Rhotrix system of equations

Let R_n be an *n*-dimensional rhotrix, *x*, the unknown *n*-dimensional rhotrix vector and *b* the right hand side rhotrix vector. The equation

$$R_n x = b \tag{7}$$

is called a *system of n rhotrix equations*. Note that in any given system of rhotrix equations the position of the non-zero elements in x and b should be the same. For instance, if in $R_5 x = b$

$$x = \left\langle \begin{matrix} & & 0 & & \\ & 0 & 0 & x_1 & \\ 0 & 0 & x_2 & 0 & 0 \\ & x_3 & 0 & 0 & \\ & & 0 & & \end{matrix} \right\rangle \text{ then we must have } b = \left\langle \begin{matrix} & & 0 & & \\ & 0 & 0 & b_1 & \\ 0 & 0 & b_2 & 0 & 0 \\ & b_3 & 0 & 0 & \\ & & 0 & & \end{matrix} \right\rangle.$$

Using the notation given in (5), rhotrix system of equation (7) gets the form $R_n \langle x^{nj} \rangle = \langle b^{nj} \rangle$. Similarly, we write a system of n linear equations in matrices, $Ax = b$ as $Ax^{nj} = b^{nj}$.

It is worthy to mention that instead of rhotrix system of equation (7), one may seek to find a solution to the minor rhotrix equation

$$R_n \langle x^{n-1k} \rangle = \langle d^{n-1k} \rangle \tag{8}$$

where R_n is an n-dimensional rhotrix, $\langle x^{n-1k} \rangle$ the unknown rhotrix vector and $\langle d^{n-1k} \rangle$ the right hand side rhotrix vector respectively with $k = 1, 2, ...t$ and $t = (n+1)/2$. This task is similar to finding a solution to (7) except that the dimension differs, therefore anything done with regards to (7) can simply be extended to (8). The following results show how (7) and similarly (8) can be solved:

Theorem 3.1. [6] *Let* $R_n = \langle a_{ij}, c_{lk} \rangle$ *be an n-dimensional rhotrix. A necessary and sufficient condition for solvability of the system* $R_n \langle x^{nj} \rangle = \langle b^{nj} \rangle$ *is that the corresponding system of equations,* $Ax^{tj} = b^{tj}$ *is solvable, where* $A = (a_{ij}) \epsilon \mathbb{R}^{t \times t}$, $x^{tj}, b^{tj} \epsilon \mathbb{R}^{t \times 1}$ *and* $t = (n+1)/2$.

Theorem 3.2. [6] *Let* $R_n = \langle a_{ij}, c_{lk} \rangle$ *be an n-dimensional rhotrix. Then the system* $R_n \langle x^{nj} \rangle = \langle b^{nj} \rangle$ *has a unique solution (or an infinite number of solutions) if and only if its corresponding system of equations* $Ax^{tj} = b^{tj}$ *has a unique solution (or an infinite number of solutions), where* $A = (a_{ij}) \epsilon \mathbb{R}^{t \times t}$, $x^{tj}, b^{tj} \epsilon \mathbb{R}^{t \times 1}$ *and* $t = (n+1)/2$.

Theorem 3.3. [6] *Let* $R_n = \langle a_{ij}, c_{lk} \rangle$ *be an n-dimensional rhotrix and* $A = (a_{ij}) \epsilon \mathbb{R}^{t \times t}$ *an embedded matrix in* R_n *where* $t = (n+1)/2$. $\langle x^{nj} \rangle$ *is a solution to the system* $R_n \langle x^{nj} \rangle = \langle b^{nj} \rangle$ *if and only if* x^{tj} *corresponding to* $\langle x^{nj} \rangle$ *is a solution to* $Ax^{tj} = b^{tj}$ *where* $x^{tj}, b^{tj} \in \mathbb{R}^{t \times 1}$.

The aim of this paper is to deal with a total rhotrix system of equations which is the task of solving (7) and (8) simultaneously. That is

$$R_n \left\langle x^{nj} \right\rangle = \left\langle b^{nj} \right\rangle$$
$$R_n \left\langle y^{n-1k} \right\rangle = \left\langle d^{n-1k} \right\rangle \quad (9)$$

where R_n, $\left\langle x^{nj} \right\rangle, \left\langle b^{nj} \right\rangle$ $\left\langle y^{n-1k} \right\rangle$ and $\left\langle d^{n-1k} \right\rangle$ are n-dimensional rhotrix and rhotrix vectors respectively with $k = 1, 2, ...t-1$ and $t = (n+1)/2$.

In [5] a special type of matrix was formed by rotating the columns of an n-dimensional rhotrix through 45° in an anticlockwise direction. This matrix is called *coupled matrix*. It was however shown in [8] that the method of converting a rhotrix in to a coupled matrix is a linear mapping.
A t-dimensional coupled matrix is of the form

$$[Ac]_t = R_n^{T/2} = \begin{pmatrix} a_{11} & & a_{12} & & \cdots & \cdots & & a_{1t} \\ & c_{11} & & c_{12} & \cdots & \cdots & c_{1t-1} & \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ & c_{t-11} & & c_{t-12} & . & . & c_{t-1t-1} & \\ a_{t1} & & a_{t2} & & \cdots & \cdots & & a_{tt} \end{pmatrix},$$

where $R_n^{T/2}$ denotes a rotation of n-dimensional rhotrix through 45° in anticlockwise direction and $t = (n+1)/2$.
Addition and multiplication (both scalar and matrix) were defined in [8] and [5] respectively. If the missing elements in a coupled matrix were filled with zeros, it can be used to find a solution to any $n \times n$ and $(n-1)\times(n-1)$ matrix system of equations simultaneously [5]. We refer the reader to [5] for a detailed description of this procedure.

Corollary 3.1. *Let* $R_n = \left\langle a_{ij}, c_{lk} \right\rangle$ *be an n-dimensional rhotrix. A necessary and sufficient condition for (9) to be solvable is that its corresponding system of equations* $Ax^{tj} = b^{tj}$ *and* $Cy^{t-1k} = d^{t-1k}$ *are solvable, where* $A = (a_{ij}) \epsilon \mathbb{R}^{t\times t}$, $x^{tj}, b^{tj} \epsilon \mathbb{R}^{t\times 1}$, $C = (c_{lk}) \epsilon \mathbb{R}^{t-1\times t-1}$, $y^{t-1k}, d^{t-1k} \epsilon \mathbb{R}^{t-1\times 1}$ with $l, k = 1, 2, ...t-1$ and $t = (n+1)/2$

Proof. The statement follows from Theorem 3.1.

Corollary 3.2. *Let* $R_n = \left\langle a_{ij}, c_{lk} \right\rangle$ *be an n-dimensional rhotrix,* $A = (a_{ij}) \epsilon \mathbb{R}^{t\times t}$ *and* $C = (c_{lk}) \epsilon \mathbb{R}^{t-1\times t-1}$ *embedded matrices in* R_n *where* $l, k = 1, 2, ...t-1$ and $t = (n+1)/2$ *. Then* $\left\langle x^{nj} \right\rangle, \left\langle y^{n-1k} \right\rangle$ *is a solution to the systems (9) if and only if* x^{tj}, y^{t-1k} *corresponding to* $\left\langle x^{nj} \right\rangle, \left\langle y^{n-1k} \right\rangle$ *is a solution to* $Ax^{tj} = b^{tj}$ $Cy^{t-1k} = d^{t-1k}$ *respectively where* $x^{tj}, b^{tj} \epsilon \mathbb{R}^{t\times 1}$, $y^{t-1k}, d^{t-1k} \epsilon \mathbb{R}^{t-1\times 1}$.

Corollary 3.3. *Let* $R_n = \left\langle a_{ij}, c_{lk} \right\rangle$ *be an n-dimensional rhotrix. Then (9) has a unique solution (or an infinite number of solutions) if and only if its corresponding system of* *equations* $Ax^{tj} = b^{tj}$ *and* $Cy^{t-1k} = d^{t-1k}$ *each has a unique solution, where* $A = (a_{ij}) \epsilon \mathbb{R}^{t\times t}$, $x^{tj}, b^{tj} \epsilon \mathbb{R}^{t\times 1}$, $C = (c_{lk}) \epsilon \mathbb{R}^{t-1\times t-1}$, $y^{t-1k}, d^{t-1k} \epsilon \mathbb{R}^{t-1\times 1}$with $l,k = 1,2,...t-1$ and $t = (n+1)/2$

Proof. The statement follows straightforwardly from Theorem 3.2.

An n-dimensional rhotrix $R_n = \langle a_{ij}, c_{lk} \rangle$ is said to be invertible if the embedded matrices $A = (a_{ij}) \epsilon \mathbb{R}^{t\times t}$ *and* $C = (c_{lk}) \epsilon \mathbb{R}^{t-1\times t-1}$ are invertible [4]. Also if the inverse of A and C are A^{-1} and C^{-1} respectively, then the inverse of R_n is $R_n^{-1} = \langle A^{-1}, C^{-1} \rangle$.

The following lemma gives a relationship between an invertible rhotrix and the existence of a unique solution to (9).

Lemma 3.1. *Let* $R_n = \langle a_{ij}, c_{lk} \rangle$ *be an n-dimensional rhotrix. Then (9) has a unique solution if and only if* R_n *is invertible.*

Proof. For a contradiction, suppose (9) has two distinct solutions say $\langle u_1^{nj} \rangle, \langle v_1^{n-1k} \rangle$ and $\langle u_2^{nj} \rangle, \langle v_2^{n-1k} \rangle$and also R_n is invertible. Since $\langle u_1^{nj} \rangle, \langle v_1^{n-1k} \rangle$ is a solution then we have

$$\begin{aligned} \langle u_1^{nj} \rangle &= I_n \langle u_1^{nj} \rangle = (R_n R_n^{-1}) \langle u_1^{nj} \rangle = R_n^{-1} (R_n \langle u_1^{nj} \rangle) = R_n^{-1} \langle b^{nj} \rangle, \\ \langle v_1^{n-1k} \rangle &= I_n \langle v_1^{n-1k} \rangle = (R_n R_n^{-1}) \langle v_1^{n-1k} \rangle = R_n^{-1} (R_n \langle v_1^{n-1k} \rangle) = R_n^{-1} \langle d^{n-1k} \rangle. \end{aligned} \tag{10}$$

Similarly, $\langle u_2^{nj} \rangle, \langle v_2^{n-1k} \rangle$ is a solution to (9) we have:

$$\begin{aligned} \langle u_2^{nj} \rangle &= I_n \langle u_2^{nj} \rangle = (R_n R_n^{-1}) \langle u_2^{nj} \rangle = R_n^{-1} (R_n \langle u_2^{nj} \rangle) = R_n^{-1} \langle b^{nj} \rangle, \\ \langle v_2^{n-1k} \rangle &= I_n \langle v_2^{n-1k} \rangle = (R_n R_n^{-1}) \langle v_2^{n-1k} \rangle = R_n^{-1} (R_n \langle v_2^{n-1k} \rangle) = R_n^{-1} \langle d^{n-1k} \rangle. \end{aligned} \tag{11}$$

From (10) and (11) we have $\langle u_1^{nj} \rangle = \langle u_2^{nj} \rangle$ and $\langle v_1^{n-1k} \rangle = \langle v_2^{n-1k} \rangle$ which is the contradiction, therefore (9) has a unique solution.

Conversely, suppose $\langle u^{nj} \rangle, \langle v^{n-1k} \rangle$ is the only solution to (9), then

$$\begin{aligned} \langle u^{nj} \rangle &= I_n \langle u^{nj} \rangle = (R_n R_n^{-1}) \langle u^{nj} \rangle = R_n^{-1} (R_n \langle u^{nj} \rangle) = R_n^{-1} \langle b^{nj} \rangle, \\ \langle v^{n-1k} \rangle &= I_n \langle v^{n-1k} \rangle = (R_n R_n^{-1}) \langle v^{n-1k} \rangle = R_n^{-1} (R_n \langle v^{n-1k} \rangle) = R_n^{-1} \langle d^{n-1k} \rangle. \end{aligned}$$

This implies that R_n is invertible

Corollary 3.4. *Let* $R_n = \langle a_{ij}, c_{lk} \rangle$ *be an n-dimensional rhotrix. If* R_n *is invertible then the only solution to (9) is* $R_n^{-1} \langle b^{nj} \rangle, R_n^{-1} \langle d^{n-1k} \rangle$.

Theorem 3.4. *Let* $R_n = \langle a_{ij}, c_{lk} \rangle$ *be an n-dimensional rhotrix. Then (9) has*
(i) a unique solution (ii) no solution, or (iii) an infinite number of solutions

Proof . In this case we only need to show that if (9) has more than one solution then it has an infinite number of solutions. Let $\langle u_1^{nj}\rangle, \langle v_1^{n-1k}\rangle$ and $\langle u_2^{nj}\rangle, \langle v_2^{n-1k}\rangle$ be two distinct solutions to (9). Then for any $\lambda \in \mathbb{R}$

$$R_n\left[\langle u_1^{nj}\rangle + \lambda\left(\langle u_1^{nj}\rangle - \langle u_2^{nj}\rangle\right)\right] = R_n\langle u_1^{nj}\rangle + \lambda\left(R_n\langle u_1^{nj}\rangle - R_n\langle u_2^{nj}\rangle\right)$$
$$= \langle b^{nj}\rangle + \lambda\left(\langle b^{nj}\rangle - \langle b^{nj}\rangle\right) = \langle b^{nj}\rangle$$

and

$$R_n\left[\langle v_1^{n-1k}\rangle + \lambda\left(\langle v_1^{n-1k}\rangle - \langle v_2^{n-1k}\rangle\right)\right] = R_n\langle v_1^{n-1k}\rangle + \lambda\left(R_n\langle v_1^{n-1k}\rangle - R_n\langle v_2^{n-1k}\rangle\right)$$
$$= \langle d^{n-1k}\rangle + \lambda\left(\langle d^{n-1k}\rangle - \langle d^{n-1k}\rangle\right) = \langle d^{n-1k}\rangle$$

Hence, for any $\lambda \in \mathbb{R}$, $\langle u_1^{nj}\rangle + \lambda\left(\langle u_1^{nj}\rangle - \langle u_2^{nj}\rangle\right), \langle v_1^{n-1k}\rangle + \lambda\left(\langle v_1^{n-1k}\rangle - \langle v_2^{n-1k}\rangle\right)$ is a solution to (9). Since the two solutions are distinct then (9) has an infinite number of solutions.

We define the homogeneous rhotrix system of equation as a system of the form:

$$\begin{aligned} R_n\langle x^{nj}\rangle &= \langle 0\rangle \\ R_n\langle y^{n-1k}\rangle &= \langle 0\rangle \end{aligned} \qquad (12)$$

where R_n is an n-dimensional rhotrix, $\langle x^{nj}\rangle$ the main rhotrix vector and $\langle y^{n-1k}\rangle$ the unknown rhotrix vector respectively with $k = 1, 2, ..., t-1$ and $t = (n+1)/2$. The homogeneous rhotrix system of equation always has a solution and the solution is $\langle x^{nj}\rangle = \langle 0\rangle$ and $\langle y^{n-1k}\rangle = \langle 0\rangle$. The following theorem gives a relationship between determinant invertible rhotrix and a nature of a solution to (12).

Theorem 3.5. *Let $R_n = \langle a_{ij}, c_{lk}\rangle$ be an n-dimensional rhotrix. The homogeneous system (12) has only trivial solution if and only if the corresponding homogeneous system of equations in matrix form $Ax^{tj} = 0$ and $Cy^{t-1k} = 0$ each has only trivial solution as a solution, where $A = (a_{ij})\epsilon\mathbb{R}^{t\times t}$, $x^{tj}, b^{tj}\epsilon\mathbb{R}^{t\times 1}$, $C = (c_{lk})\epsilon\mathbb{R}^{t-1\times t-1}$, $y^{t-1k}\epsilon\mathbb{R}^{t-1\times 1}$ with* $l, k = 1, 2, ...t-1$ and $t = (n+1)/2$

For an n-dimensional rhotrix, R_n, and $A = (a_{ij})\epsilon\mathbb{R}^{t\times t}$, $C = (c_{lk})\epsilon\mathbb{R}^{t-1\times t-1}$ the embedded matrices in R_n the determinant of R_n denoted as $\det(R_n)$ is defined [4] as follows:

$$\det(R_n) = \det(A)\det(C).$$

Theorem 3.6. *Let $R_n = \langle a_{ij}, c_{lk}\rangle$ be an n-dimensional rhotrix. Then the following three statements are equivalent*

(i) R_n *is invertible*

(ii) *The homogeneous rhorix system of equations (12) has only trivial solution as a solution*

(iii) $\det(R_n) \neq 0$

Proof. Suppose R_n is invertible then the embedded matrices $A = (a_{ij}) \epsilon \mathbb{R}^{t\times t}$ *and* $C = (c_{lk}) \epsilon \mathbb{R}^{t-1\times t-1}$ are invertible. If A is invertible then it is row equivalent to I_t [10,11,12], similarly C is invertible then it is row equivalent to I_{t-1}. Since $\det(I_t) \neq 0$ and $\det(I_{t-1}) \neq 0$ then $\det(A) \neq 0$ and $\det(C) \neq 0$, consequently $\det(R_n) \neq 0$. Conversely, if R_n is not invertible then either A is not invertible or C is not invertible or both. If A is not invertible then it is row equivalent to a matrix with zero row [10,11,12] and hence $\det(A) = 0$ which implies that $\det(R_n) = 0$. A similar argument can be used for the other two cases and show that in each case $\det(R_n) = 0$.
It follows that (i) is equivalent to (iii)

If (12) has trivial solution as the only solution then by Theorem 3.5 each of $Ax^{tj} = 0$ and $Cx^{t-1k} = 0$ each has only trivial solution as a solution. If each of the system $Ax^{tj} = 0$ and $Cx^{t-1k} = 0$ has trivial solution only then A and C are row equivalent to I_t and I_{t-1} respectively. Therefore both A and C are invertible, consequently R_n is invertible. Suppose R_n is invertible with inverse $R_n^{-1} = \langle A^{-1}, C^{-1} \rangle$ then it follows that

$$\langle x^{nj} \rangle = I_n \langle x^{nj} \rangle = \left(R_n^{-1} R_n\right) \langle x^{nj} \rangle = R_n^{-1} \left(R_n \langle x^{nj} \rangle\right) = R_n^{-1} \langle 0^{nj} \rangle = \langle 0^{nj} \rangle$$
$$\langle y^{n-1k} \rangle = I_n \langle y^{n-1k} \rangle = \left(R_n^{-1} R_n\right) \langle y^{n-1k} \rangle = R_n^{-1} \left(R_n \langle y^{n-1k} \rangle\right) = R_n^{-1} \langle 0^{n-1k} \rangle = \langle 0^{n-1k} \rangle$$

is the only solution to (10). Hence (i) is equivalent to (ii) and the theorem statements now follow.

4. An example

Find a solution to the following rhotrix system of equation (if it exists)

$$R_5 \langle x^{52} \rangle = \langle b^{52} \rangle$$
$$R_5 \langle y^{42} \rangle = \langle d^{42} \rangle$$

where

$$R_5 = \left\langle \begin{matrix} & & 0 & & \\ & 2 & 1 & 2 & \\ 3 & 1 & 4 & 1 & 4 \\ & 3 & -1 & 2 & \\ & & 1 & & \end{matrix} \right\rangle, \langle x^{52} \rangle = \left\langle \begin{matrix} & & 0 & & \\ & 0 & 0 & x_1 & \\ 0 & 0 & x_2 & 0 & 0 \\ & x_3 & 0 & 0 & \\ & & 0 & & \end{matrix} \right\rangle,$$

$$\langle y^{42} \rangle = \left\langle \begin{matrix} & & 0 & & \\ & 0 & 0 & 0 & \\ 0 & 0 & 0 & y_1 & 0 \\ & 0 & y_2 & 0 & \\ & & 0 & & \end{matrix} \right\rangle, \langle b^{52} \rangle = \left\langle \begin{matrix} & & 0 & & \\ & 0 & 0 & 6 & \\ 0 & 0 & 8 & 0 & 0 \\ & 10 & 0 & 0 & \\ & & 0 & & \end{matrix} \right\rangle$$

$$\text{and } \langle d^{42} \rangle = \left\langle \begin{array}{ccccc} & & 0 & & \\ & 0 & 0 & 0 & \\ 0 & 0 & 0 & 3 & 0 \\ & 0 & 1 & 0 & \\ & & 0 & & \end{array} \right\rangle$$

It is easy to verify that the corresponding system of equations $Ax^{32} = b^{32}$ and $Cy^{22} = d^{22}$ are

$$Ax^{32} = b^{32} = \begin{bmatrix} 0 & 2 & 4 \\ 2 & 4 & 2 \\ 3 & 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 8 \\ 10 \end{bmatrix} \text{ and}$$

$$Cy^{22} = d^{22} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

respectively. The solutions to the systems are

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

Therefore, by Corollary 3.2 the problem has a solution which is

$$\langle x^{52} \rangle = \left\langle \begin{array}{ccccc} & & 0 & & \\ & 0 & 0 & 1 & \\ 0 & 0 & 1 & 0 & 0 \\ & 1 & 0 & 0 & \\ & & 0 & & \end{array} \right\rangle, \langle y^{42} \rangle = \left\langle \begin{array}{ccccc} & & 0 & & \\ & 0 & 0 & 0 & \\ 0 & 0 & 0 & 2 & 0 \\ & 0 & 1 & 0 & \\ & & 0 & & \end{array} \right\rangle.$$

Conclusion

As an extension to the idea of rhotrix system of equations discussed in [6], in this paper rhotrix system of equation which consists of all the possible equations was discussed and a number of solvability conditions were suggested. It was however shown that the task of solving this type of problem leads to solving $n \times n$ and $(n-1)\times(n-1)$ matrix system of equations simultaneously. The method of solving this matrix system of equations was suggested in [5] which involved the use of coupled matrix operator introduced in [8]. Therefore, our work highlighted the importance of a coupled matrix and coupled matrix operator in solving systems of rhotrix equations.

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