A determinant method for solving rhotrix system of equations

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Abstract

The determinant method (Cramer rule) is one of the well-known methods that is formulated and proved in linear algebra on matrices. In this paper we extend this method to the concept of rhotrix. Rhotrix is an object that lies in some way between $n \times n$ dimensional matrices and $(2n-1)\times(2n-1)$ dimensional matrices and representation of vectors in rhotrix is different from the representation of vectors in matrix.

Keywords: determinant, Cramer's rule; rhotrix, rhotrix vectors

AMS Subject Classifications [2000]: 15A06, 15A15

1. Introduction

The concept of rhotrix was first introduced by Ajibade [1] as an extension of the initiative on matrix-tertions and matrix-noitrets suggested by Atanassov and Shannon [2]. The initial algebra and analysis of rhotrices was presented in [1]. The multiplication of rhotrices defined by Ajibade [1] is as follows: Let R and Q be two rhotrices such that

$$R = \left\langle \begin{matrix} & a & \\ b & h(R) & d \\ & e & \end{matrix} \right\rangle \text{ and } Q = \left\langle \begin{matrix} & f & \\ g & h(Q) & j \\ & k & \end{matrix} \right\rangle. \tag{1}$$

The addition and multiplication of rhotrices R and Q defined by Ajibade [1] are as follows:

$$R + Q = \left\langle \begin{matrix} & a+f & \\ b+g & h(R)+h(Q) & d+j \\ & e+k & \end{matrix} \right\rangle,$$

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$$R \circ Q = \left\langle \begin{array}{ccc} & ah(Q)+fh(R) & \\ bh(Q)+gh(R) & h(R)h(Q) & dh(Q)+jh(R) \\ & eh(Q)+kh(R) & \end{array} \right\rangle.$$

Another multiplication method for rhotrices called *row-column multiplication* was introduced by Sani [3] in an effort to answer some questions raised by Ajibade [1]. The row-column multiplication method is in a similar way as that of multiplication of matrices and is illustrated using the matrices R and Q defined in (1) as:

$$R \circ Q = \left\langle \begin{array}{ccc} & af+dg & \\ bf+eg & h(R)h(Q) & aj+dk \\ & bj+ek & \end{array} \right\rangle.$$

A generalization of the row-column multiplication method for n-dimensional rhotrices was given by Sani [4]. That is: given n-dimensional rhotrices $R_n = \langle a_{ij}, c_{lk} \rangle$ and $Q_n = \langle b_{ij}, d_{lk} \rangle$ the multiplication of R_n and Q_n is as follows:

$$R_n \circ Q_n = \langle a_{i_1 j_1}, c_{l_1 k_1} \rangle \circ \langle b_{i_2 j_2}, d_{l_2 k_2} \rangle = \left\langle \sum_{i_2 j_1 = 1}^{t} (a_{i_1 j_1} b_{i_2 j_2}), \sum_{l_2 k_1 = 1}^{t-1} (c_{l_1 k_1} d_{l_2 k_2}) \right\rangle,\ t = (n+1)/2.$$

The method of converting a rhotrix to a special matrix called 'coupled matrix' was suggested by Sani [5]. This idea was used to solve systems of $n \times n$ and $(n-1)\times(n-1)$ matrix problems simultaneously. The concept of vectors, one-sided system of equations and eigenvector eigenvalue problem in rhotrices were introduced by Aminu [6]. A necessary and sufficient condition for the solvability of one sided system of rhotrix was also presented in [6]. If a system is solvable it was shown how a solution can be found. Rhotrix vector spaces and their properties were presented by Aminu [7]. Linear mappings and square root of a rhotrix were discussed by Aminu in [8] and [9] respectively. The rhotrix system of equations was discussed by Aminu [10] and a number of solvability conditions were suggested.

To the author's knowledge Cramer rule was not extended to rhotrix. It is the primary aim of this paper to extend this well-known theorem to rhotrix and show how it can be used to solve rhotrix system of equations.

2. Rhotrix and its basic properties

Let $t = (n+1)/2$ for $n \in \mathbb{N}$. By 'rhotrix' we understand an object that lies in some way between $n \times n$ dimensional matrices and $(2n-1)\times(2n-1)$ dimensional matrices. That is an n-dimensional rhotrix is the following:

$$R_n = \langle a_{ij}, c_{lk} \rangle = \left\langle \begin{array}{ccccccc} & & & a_{11} & & & \\ & & a_{21} & c_{11} & a_{12} & & \\ & a_{31} & c_{21} & a_{22} & c_{12} & a_{13} & \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{t1} & \ldots & \ldots & \ldots & \ldots & \ldots & a_{1t} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ & a_{tt-2} & c_{t-1t-2} & a_{t-1t-1} & c_{t-2t-1} & a_{t-2\,t} & \\ & & a_{t\,t-1} & c_{t-1\,t-1} & a_{t-1\,t} & & \\ & & & a_{tt} & & & \end{array} \right\rangle, \tag{2}$$

where $a_{ij}, a_{lk} \epsilon \mathbb{R}$ for $i, j = 1, 2, ..., t$ and $k, l = 1, 2..., t-1$. It is straightforward to verify that the addition of n-dimensional rhotrices $R_n = \langle a_{ij}, c_{lk} \rangle$ and $Q_n = \langle b_{ij}, d_{lk} \rangle$ is

$$R_n + Q_n = \langle a_{ij}, c_{lk} \rangle + \langle b_{ij}, d_{lk} \rangle = \langle (a_{ij} + b_{ij}), (c_{lk} + d_{lk}) \rangle, \tag{3}$$

where $i, j = 1, 2, ..., t$ and $l, k = 1, 2...t-1$ with $t = (n+1)/2$.

We will use throughout this paper the row-column multiplication method of rhotrices.

Rhotrix vectors (either row vectors or column vectors) can be represented in t different ways where $t = (n+1)/2$. This is different compared to vectors in matrices that can be represented in a unique way. For more information on rhotrix vectors the reader is referred to [6] and [7].

There is a unique representation of any t-dimensional matrix vector while any n-dimensional rhotrix vector can be represented in t different ways where $t = (n+1)/2$. This can be illustrated as follows: A 3-dimensional matrix column vector is uniquely given as

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

whereas, a 5-dimensional rhotrix column vector could be any of

$$\left\langle \begin{array}{ccccc} & & x_1 & & \\ & x_2 & 0 & 0 & \\ x_3 & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & \\ & & 0 & & \end{array} \right\rangle, \left\langle \begin{array}{ccccc} & & 0 & & \\ & 0 & 0 & x_1 & \\ 0 & 0 & x_2 & 0 & 0 \\ & x_3 & 0 & 0 & \\ & & 0 & & \end{array} \right\rangle \text{ or } \left\langle \begin{array}{ccccc} & & 0 & & \\ & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & x_1 \\ & 0 & 0 & x_2 & \\ & & x_3 & & \end{array} \right\rangle. \tag{4}$$

We use the notation introduced in [6] as

$$\langle x^{nj} \rangle \tag{5}$$

to represent the main rhotrix column vector and the main rhotrix row vector is denoted by

$$\langle x^{in} \rangle \tag{6}$$

where $i, j = 1, 2, ..., t$ with $t = (n+1)/2$. Thus, the three rhotrix column vectors in (4) are $\langle x^{51} \rangle, \langle x^{52} \rangle$ and $\langle x^{53} \rangle$ respectively. Similarly, we denote the other columns and rows which are not the main as

$$\langle x^{n-1k} \rangle$$

and

$$\langle x^{ln-1} \rangle$$

respectively, where $k, l = 1, 2, ..., t-1$. Consequently, the *n*-dimensional matrix column vectors will be represented as x^{nj}. The *n*-dimensional identity rhotrix will be denoted by I_n and is given by

$$I_n = \left\langle \begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & 1 & 0 & & \\ & 0 & 0 & 1 & 0 & 0 & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ & 0 & 0 & 1 & 0 & 0 & \\ & & 0 & 1 & 0 & & \\ & & & 1 & & & \end{array} \right\rangle.$$

We also denote by 0 the usual zero, which is the neutral element under addition and for convenience we use the same symbol to denote any rhotrix or rhotrix vector whose every component is 0.

We will now summarize some basic properties of rhotrices that will be used later on. The following properties hold for *n*-dimensional rhotrices A, B and C over $\mathbb{R}$ and $\alpha \in \mathbb{R}$:

$$A + 0 = 0 + A = A$$

$$A + B = B + A$$

$$(A+B) + C = A + (B+C)$$

$$\alpha(A+B) = \alpha A + \alpha B$$

$$A(B+C) = AB + AC$$

$$A(BC) = (AB)C$$

$$AI_n = A = I_n A$$

3. Rhotrix system of equations

Let R_n be an *n*-dimensional rhotrix, *x*, the unknown *n*-dimensional rhotrix vector and *b* the right hand side rhotrix vector. The equation

$$R_n x = b \tag{7}$$

is called a *system of n rhotrix equations*. Note that in any given system of rhotrix equations the position of the non-zero elements in x and b should be the same. For instance, if in $R_5 x = b$

$$x = \left\langle \begin{array}{ccccc} & & 0 & & \\ & 0 & 0 & x_1 & \\ 0 & 0 & x_2 & 0 & 0 \\ & x_3 & 0 & 0 & \\ & & 0 & & \end{array} \right\rangle \text{ then we must have } b = \left\langle \begin{array}{ccccc} & & 0 & & \\ & 0 & 0 & b_1 & \\ 0 & 0 & b_2 & 0 & 0 \\ & b_3 & 0 & 0 & \\ & & 0 & & \end{array} \right\rangle .$$

Using the notation given in (5), system of n rhotrix equation (7) gets the form $R_n \langle x^{nj} \rangle = \langle b^{nj} \rangle$. Similarly, we write a system of n linear equations in matrices, $Ax = b$ as $Ax^{nj} = b^{nj}$.
Instead of rhotrix equation (7), one may seek to find a solution to the minor rhotrix equation

$$R_n \langle x^{n-1k} \rangle = \langle d^{n-1k} \rangle \tag{8}$$

where R_n is an n-dimensional rhotrix, $\langle x^{n-1k} \rangle$ the unknown rhotrix vector and $\langle d^{n-1k} \rangle$ the right hand side rhotrix vector respectively with $k = 1, 2, \ldots t$ and $t = (n+1)/2$. This task is similar to finding a solution to (7) except that the dimension differs, therefore anything done with regards to (7) can simply be applied to (8). In this paper we will deal with a total rhotrix system of equations which is the task of solving (7) and (8) simultaneously using determinant method. That is

$$\begin{aligned} R_n \langle x^{nj} \rangle &= \langle b^{nj} \rangle \\ R_n \langle y^{n-1k} \rangle &= \langle d^{n-1k} \rangle \end{aligned} \tag{9}$$

where R_n, $\langle x^{nj} \rangle, \langle b^{nj} \rangle$ $\langle x^{n-1k} \rangle$ and $\langle d^{n-1k} \rangle$ are as defined in (7) and (8) with $k = 1, 2, \ldots t$ and $t = (n+1)/2$.
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Theorem 3.1. [10] *Let $R_n = \langle a_{ij}, c_{lk} \rangle$ be an n-dimensional rhotrix. A necessary and sufficient condition for (9) to be solvable is that its corresponding system of equations $Ax^{tj} = b^{tj}$ and $Cy^{t-1k} = d^{t-1k}$ are solvable, where $A = (a_{ij}) \epsilon \mathbb{R}^{t \times t}$, $x^{tj}, b^{tj} \epsilon \mathbb{R}^{t \times 1}$, $C = (c_{lk}) \epsilon \mathbb{R}^{t-1 \times t-1}$, $y^{t-1k}, d^{t-1k} \epsilon \mathbb{R}^{t-1 \times 1}$* with $l, k = 1, 2, \ldots t-1$ and $t = (n+1)/2$.

Theorem 3.2. [10] *Let $R_n = \langle a_{ij}, c_{lk} \rangle$ be an n-dimensional rhotrix, $A = (a_{ij}) \epsilon \mathbb{R}^{t \times t}$ and $C = (c_{lk}) \epsilon \mathbb{R}^{t-1 \times t-1}$, embedded matrices in R_n where* $l, k = 1, 2, \ldots t-1$ and $t = (n+1)/2$. *Then $\langle x^{nj} \rangle, \langle y^{n-1k} \rangle$ is a solution to the systems (9) if* *and only if* x^{tj}, y^{t-1k} *corresponding to* $\langle x^{nj}\rangle, \langle y^{n-1k}\rangle$ *is a solution to* $Ax^{tj} = b^{tj}$ $Cy^{t-1k} = d^{t-1k}$ *respectively where* $x^{tj}, b^{tj} \epsilon \mathbb{R}^{t\times 1}$, $y^{t-1k}, d^{t-1k} \epsilon \mathbb{R}^{t-1\times 1}$.

It follows from Theorems 3.1 and 3.2 that (9) can be solved using the idea of coupled matrix introduced in [5]. This was mentioned in [10]

Theorem 3.3. [10] *Let* $R_n = \langle a_{ij}, c_{lk}\rangle$ *be an n-dimensional rhotrix. Then (9) has a unique solution (or an infinite number of solutions) if and only if its corresponding system of equations* $Ax^{tj} = b^{tj}$ *and* $Cy^{t-1k} = d^{t-1k}$ *each has a unique solution, where* $A = (a_{ij}) \epsilon \mathbb{R}^{t\times t}$, $x^{tj}, b^{tj} \epsilon \mathbb{R}^{t\times 1}$, $C = (c_{lk}) \epsilon \mathbb{R}^{t-1\times t-1}$, $y^{t-1k}, d^{t-1k} \epsilon \mathbb{R}^{t-1\times 1}$ with $l, k = 1, 2, \ldots t-1$ and $t = (n+1)/2$

An n-dimensional rhotrix $R_n = \langle a_{ij}, c_{lk}\rangle$ is said to be invertible if the embedded matrices $A = (a_{ij}) \epsilon \mathbb{R}^{t\times t}$ and $C = (c_{lk}) \epsilon \mathbb{R}^{t-1\times t-1}$ are invertible [3]. Also if the inverse of A and C are A^{-1} and C^{-1} respectively, then the inverse of R_n is $R_n^{-1} = \langle A^{-1}, C^{-1}\rangle$.

Lemma 3.1. [10] *Let* $R_n = \langle a_{ij}, c_{lk}\rangle$ *be an n-dimensional rhotrix. Then (9) has a unique solution if and only if* R_n *is invertible and in this case the solution is* $\langle x^{nj}\rangle = R_n^{-1}\langle b^{nj}\rangle, \langle y^{n-1k}\rangle = R_n^{-1}\langle d^{n-1k}\rangle$.

4. Cramer's rule on rhotrix

Let $R_n = \langle a_{ij}, c_{lk}\rangle$ be an n-dimensional rhotrix defined in (2), the determinant of $R_n = \langle a_{ij}, c_{lk}\rangle$ is defined [4] as

$$\det(R_n) = \det\left(\langle a_{ij}, c_{lk}\rangle\right) = \det(A)\det(C) \tag{10}$$

where $A = (a_{ij}) \epsilon \mathbb{R}^{t\times t}$ and $C = (c_{lk}) \epsilon \mathbb{R}^{t-1\times t-1}$ are the embedded matrices in R_n .
Consider the rhotrix equation $R_n\langle x^{nj}\rangle = \langle b^{nj}\rangle$ where R_n is an n-dimensional rhotrix, $\langle x^{nj}\rangle$ the unknown n-dimensional rhotrix vector and $\langle b^{nj}\rangle$ the right hand side rhotrix vector. Let $R_n^{A_i}$ be a rhotrix formed by replacing the i^{th} column of the matrix $A = (a_{ij}) \epsilon \mathbb{R}^{t\times t}$ embedded in R_n by the non-zero column of the rhotrix column vector $\langle b^{nj}\rangle$. Similarly, denote by $R_n^{C_i}$ a rhotrix formed by replacing the i^{th} column of the matrix $C = (c_{lk}) \epsilon \mathbb{R}^{t-1\times t-1}$ embedded in R_n by the non-zero column of the rhotrix column vector $\langle d^{n-1k}\rangle$.Furthermore, denote by

$$D=\det(R_n),\ M_1=\det\left(R_n^{A_1}\right),\ M_2=\det\left(R_n^{A_2}\right),...,M_t=\det\left(R_n^{A_t}\right)$$
$$N_1=\det\left(R_n^{C_1}\right),\ N_2=\det\left(R_n^{C_2}\right),...,N_{t-1}=\det\left(R_n^{C_{t-1}}\right)$$
where $t=(n+1)/2$.

Theorem 4.1. (Cramer) *Let $R_n=\langle a_{ij},c_{lk}\rangle$ be an n-dimensional rhotrix. The system (9) has a unique solution if and only if $D\neq 0$. Moreover, the x_i and y_i components of the solution are given by*

$$x_1=\frac{M_1}{D},\ x_2=\frac{M_2}{D},..,x_t=\frac{M_t}{D}$$
$$y_1=\frac{N_1}{D},\ y_2=\frac{N_2}{D},..,y_{t-1}=\frac{N_{t-1}}{D}$$
where $i,j=1,2,...,t$ and $l,k=1,2,...,t-1$ with $t=(n+1)/2$.

Proof. Suppose (9) has a unique solution then it follows from Theorem 3.3 that the corresponding system of equations, $Ax^{tj}=b^{tj}$ and $Cx^{t-1k}=d^{t-1k}$ each has a unique solution, where $A=(a_{ij})\epsilon\mathbb{R}^{t\times t}$, $x^{tj},b^{tj}\epsilon\mathbb{R}^{t\times 1}$ and $t=(n+1)/2$. It follows from [9] that $Ax^{tj}=b^{tj}$ and $Cx^{t-1k}=d^{t-1k}$ each has a unique solution if and only if A and also C are invertible, moreover A and C are invertible if and only if $\det(A)\neq 0$ and $\det(C)\neq 0$ respectively [11,12,13]. Because $\det(A)\neq 0$ and $\det(C)\neq 0$ then $D\neq 0$.
Conversely, suppose $D\neq 0$, then $\det(A)\neq 0$ and $\det(C)\neq 0$ the statement now follows from the Cramer's rule on matrices and Theorems 3.1, 3.2 and 3.3.

5. An example

Use determinant method (Cramer's rule) to solve the rhotrix system of equation

$$R_5\left\langle x^{52}\right\rangle=\left\langle b^{52}\right\rangle$$
$$R_5\left\langle y^{42}\right\rangle=\left\langle d^{42}\right\rangle$$

where

$$R_5=\left\langle\begin{array}{ccccc} & & 1 & & \\ & 1 & 3 & 1 & \\ 3 & 1 & -2 & 2 & 1 \\ & 1 & 2 & -3 & \\ & & -1 & & \end{array}\right\rangle,\left\langle x^{52}\right\rangle=\left\langle\begin{array}{ccccc} & & 0 & & \\ & 0 & 0 & x_1 & \\ 0 & 0 & x_2 & 0 & 0 \\ & x_3 & 0 & 0 & \\ & & 0 & & \end{array}\right\rangle,$$

$$\left\langle b^{52}\right\rangle=\left\langle\begin{array}{ccccc} & & 0 & & \\ & 0 & 0 & 5 & \\ 0 & 0 & -1 & 0 & 0 \\ & 3 & 0 & 0 & \\ & & 0 & & \end{array}\right\rangle,\left\langle y^{42}\right\rangle=\left\langle\begin{array}{ccccc} & & 0 & & \\ & 0 & 0 & 0 & \\ 0 & 0 & 0 & y_1 & 0 \\ & 0 & y_2 & 0 & \\ & & 0 & & \end{array}\right\rangle$$

$$\text{and } \left\langle d^{42}\right\rangle=\left\langle\begin{array}{ccccc} & & 0 & & \\ & 0 & 0 & 0 & \\ 0 & 0 & 0 & -2 & 0 \\ & 0 & 2 & 0 & \\ & & 0 & & \end{array}\right\rangle.$$

Using the definition of determinant of rhotrix we have

$$D=\det(R_5)=\det\left(\left\langle\begin{array}{ccccc} & & 1 & & \\ & 1 & 3 & 1 & \\ 3 & 1 & -2 & 2 & 1 \\ & 1 & 2 & -3 & \\ & & -1 & & \end{array}\right\rangle\right)=20, M_1=\det\left(\left\langle\begin{array}{ccccc} & & 5 & & \\ & -1 & 3 & 1 & \\ 3 & 1 & -2 & 2 & 1 \\ & 1 & 2 & -3 & \\ & & -1 & & \end{array}\right\rangle\right)=80$$

$$M_2=\det\left(\left\langle\begin{array}{ccccc} & & 1 & & \\ & 1 & 3 & 5 & \\ 3 & 1 & -1 & 2 & 1 \\ & 3 & 2 & -3 & \\ & & -1 & & \end{array}\right\rangle\right)=-40, \text{and } M_3=\det\left(\left\langle\begin{array}{ccccc} & & 1 & & \\ & 1 & 3 & 1 & \\ 3 & 1 & -2 & 2 & 5 \\ & 1 & 2 & -1 & \\ & & 3 & & \end{array}\right\rangle\right).$$

Also,

$$N_1=\det\left(\left\langle\begin{array}{ccccc} & & 5 & & \\ & -1 & -2 & 1 & \\ 3 & 2 & -2 & 2 & 1 \\ & 1 & 2 & -3 & \\ & & -1 & & \end{array}\right\rangle\right)=-40 \text{ and } N_2=\det\left(\left\langle\begin{array}{ccccc} & & 5 & & \\ & -1 & 3 & 1 & \\ 3 & 1 & -2 & -2 & 1 \\ & 1 & 2 & -3 & \\ & & -1 & & \end{array}\right\rangle\right)=40$$

Thus we have,

$$x_1=\frac{M_1}{D}=\frac{80}{20}=4,\ x_2=\frac{M_2}{D}=\frac{-40}{20}=-2 \text{ and } x_3=\frac{M_3}{D}=\frac{60}{20}=3$$
$$y_1=\frac{N_1}{D}=\frac{-40}{20}=-2 \text{ and } \mathrm{y}_2=\frac{N_2}{D}=\frac{40}{20}=2.$$

Hence, the solution to the system are the vectors

$$\langle x^{52} \rangle = \left\langle \begin{array}{ccccc} & & 0 & & \\ & 0 & 0 & 4 & \\ 0 & 0 & -2 & 0 & 0 \\ & 3 & 0 & 0 & \\ & & 0 & & \end{array} \right\rangle \text{ and } \langle y^{42} \rangle = \left\langle \begin{array}{ccccc} & & 0 & & \\ & 0 & 0 & 0 & \\ 0 & 0 & 0 & -2 & 0 \\ & 0 & 2 & 0 & \\ & & 0 & & \end{array} \right\rangle .$$

Conclusion

In this paper the well-known determinant method (Cramer's rule) was extended to rhotrix system of equations. The method was successfully formulated and proved in rhotrix and also shows how the method can easily be used to find a solution to rhotrix system of equations (if it exists). Unlike in matrix system of equations where it is required that the matrix must be square in order to apply the Cramer's rule it is even more stronger in rhotrix where it can be applied on any rhotrix system of equations provided that determinant of the given rhotrix is not zero.

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