CERTAIN QUADRATIC EXTENSIONS

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S. M. TUDUNKAYA[1] AND S. O. MAKANJUOLA[2]

Kano University of Science and Technology, Wudil, Nigeria[1]

University of Ilorin, Ilorin, Nigeria[2]

E-mail: *tudunkayaunique@yahoo.com*[1] *somakanjuola@unilorin.edu.ng*[2]

Abstract

In this piece of note, a particular finite field construction was explored; through the use of a rather new Mathematical object named rhotrix with the operations defined on it in Ajibade [1]. Some examples were provided to further buttress the method, the hope is that the note may be useful and interesting lecture note.

Key words: Field; Vector space; Finite field; Field extension

1 Introduction

The algebra of rhotrices began in 2003, when Ajibade [1], defined an object of the set

$$R = \left\{ \left\langle \begin{matrix} & a & \\ b & c & d \\ & e & \end{matrix} \right\rangle : a, b, c, d, e \in \Re \right\}$$

as rhotrix, as a result of its rhomboid nature. The entry at the perpendicular intersection of the two diagonals of a rhotrix R denoted by $h(R)$, was defined as heart (that is c in the above definition).

$$R + Q = \left\langle \begin{matrix} & a & \\ b & h(R) & d \\ & e & \end{matrix} \right\rangle + \left\langle \begin{matrix} & f & \\ g & h(Q) & j \\ & k & \end{matrix} \right\rangle = \left\langle \begin{matrix} & a+f & \\ b+g & h(R)+h(Q) & d+j \\ & e+k & \end{matrix} \right\rangle$$

was defined as the addition of two rhotrices and $-A$ was given as the additive inverse of A, since

$$A + (-A) = \left\langle \begin{matrix} & a & \\ b & h(R) & d \\ & e & \end{matrix} \right\rangle + \left\langle \begin{matrix} & -a & \\ -b & -h(R) & -d \\ & -e & \end{matrix} \right\rangle = \left\langle \begin{matrix} & 0 & \\ 0 & 0 & 0 \\ & 0 & \end{matrix} \right\rangle$$

which was the additive identity of R. It was shown that $\{R, +\} \cup 0$ where $0 = \left\langle \begin{matrix} & 0 & \\ 0 & 0 & 0 \\ & 0 & \end{matrix} \right\rangle$ is a commutative group. Scalar multiplication was defined as follows:

$$\alpha R = \alpha \left\langle \begin{array}{ccc} & a & \\ b & h(R) & d \\ & e & \end{array} \right\rangle = \left\langle \begin{array}{ccc} & \alpha a & \\ \alpha b & \alpha h(R) & \alpha d \\ & \alpha e & \end{array} \right\rangle$$

Also, the following multiplication method was given:

$$R \bullet Q = \left\langle \begin{array}{ccc} & a & \\ b & h(R) & d \\ & e & \end{array} \right\rangle \bullet \left\langle \begin{array}{ccc} & f & \\ g & h(Q) & j \\ & k & \end{array} \right\rangle = \left\langle \begin{array}{ccc} & ah(Q)+fh(R) & \\ bh(Q)+gh(R) & h(R)h(Q) & dh(Q)+jh(R) \\ & eh(Q)+kh(R) & \end{array} \right\rangle$$

The set R was proved to be a commutative algebra. The multiplicative identity of R was defined as:

$$I = \left\langle \begin{array}{ccc} & 0 & \\ 0 & 1 & 0 \\ & 0 & \end{array} \right\rangle$$

If $$R \bullet Q = \left\langle \begin{array}{ccc} & a & \\ b & h(R) & d \\ & e & \end{array} \right\rangle \bullet \left\langle \begin{array}{ccc} & f & \\ g & h(Q) & j \\ & k & \end{array} \right\rangle = \left\langle \begin{array}{ccc} & 0 & \\ 0 & 1 & 0 \\ & 0 & \end{array} \right\rangle$$

Then $$Q = R^{-1} = -\frac{1}{h(R)^2} \left\langle \begin{array}{ccc} & a & \\ b & -h(R) & d \\ & e & \end{array} \right\rangle, \quad h(P) \neq 0$$

In Mohammed [2], a generalised definition of a rhotrix R of dimension n with the operations defined above, was presented as the set

$$A(n) = \left\{ \left\langle \begin{array}{ccccccc} & & & a_1 & & & \\ & & a_2 & a_3 & a_4 & & \\ & \dots & \dots & \dots & \dots & \dots & \\ a_{\left\{\frac{(t+1)}{2}\right\}-\frac{n}{2}} & \dots & \dots & a_{\left\{\frac{(t+1)}{2}\right\}} & \dots & \dots & a_{\left\{\frac{(t+1)}{2}\right\}+\frac{n}{2}} \\ & \dots & \dots & \dots & \dots & \dots & \\ & & a_{t-3} & a_{t-2} & a_{t-1} & & \\ & & & a_t & & & \end{array} \right\rangle a_i \epsilon \Re \right\}$$

where $t = \frac{(n^2+1)}{2}$, $n \epsilon 2Z^+ + 1$ and $\frac{n}{2}$ is the integer value upon division of n by 2.

2 Rhotrix quadratic extensions

Recall that a polynomial of degree 2 is called a quadratic expression, because the highest power in it is 2. The method of extension illustrated in Joyner [3], and the discussions of concepts in Lang [4] were applied here. Also, by Tudunkaya and Makanjuola (*submitted*), if

$$F_p[R] = \left\{ \left\langle \begin{array}{ccccccc} & & & 0_1 & & & \\ & & 0_2 & 0_3 & 0_4 & & \\ & \cdots & \ldots & \ldots & \ldots & \cdots & \\ 0_{\alpha} & \cdots & \ldots & a_\beta & \ldots & \cdots & 0_\pi \\ & \cdots & \ldots & \ldots & \ldots & \cdots & \\ & & 0_{t-3} & 0_{t-2} & 0_{t-1} & & \\ & & & a_t & & & \end{array} \right\rangle \forall\, a \in Z_p \right\}$$

$\alpha = \left\{\frac{(t+1)}{2}\right\} - \frac{n}{2}$, $\beta = \left\{\frac{(t+1)}{2}\right\}$, $\pi = \left\{\frac{(t+1)}{2}\right\} + \frac{n}{2}$ and $(F_p[R], +, \bullet)$ where $(+)$ and $(\bullet)$ denote addition and multiplication of rhotrices is a field. Suppose $N \epsilon F_p[R]$ and there exist no element $M \epsilon F_p[R]$ such that $N = M^2$, let

$$B_1 = \left\langle \begin{array}{ccccccc} & & & 0_1 & & & \\ & & 0_2 & 0_3 & 0_4 & & \\ & \cdots & \ldots & \ldots & \ldots & \cdots & \\ 0_{\alpha} & \cdots & \ldots & 1_\beta & \ldots & \cdots & 0_\pi \\ & \cdots & \ldots & \ldots & \ldots & \cdots & \\ & & 0_{t-3} & 0_{t-2} & 0_{t-1} & & \\ & & & a_t & & & \end{array} \right\rangle$$

and

$$B_2 = \sqrt{ \begin{array}{c} 0_1 \\ \left\langle \begin{array}{ccccccc} & & 0_2 & 0_3 & 0_4 & & \\ & \cdots & \ldots & \ldots & \ldots & \cdots & \\ 0_{\alpha} & \cdots & \ldots & a_\beta & \ldots & \cdots & 0_\pi \\ & \cdots & \ldots & \ldots & \ldots & \cdots & \\ & & 0_{t-3} & 0_{t-2} & 0_{t-1} & & \end{array} \right\rangle \\ a_t \end{array} }$$

such that

$$B_2{}^2 = \left\langle \begin{array}{ccccccc} & & & 0_1 & & & \\ & & 0_2 & 0_3 & 0_4 & & \\ & \cdots & \ldots & \ldots & \ldots & \cdots & \\ 0_{\alpha} & \cdots & \ldots & a_\beta & \ldots & \cdots & 0_\pi \\ & \cdots & \ldots & \ldots & \ldots & \cdots & \\ & & 0_{t-3} & 0_{t-2} & 0_{t-1} & & \\ & & & a_t & & & \end{array} \right\rangle$$

Let the set $F[R] = \{XB_1 + YB_2 : X, Y \epsilon F_p[R]\}$, define (+) on $F[R]$ to be rhotrix addition component wise $mod\ p$ and (•) of any two elements $\left(X_1 + X_2\sqrt{N}\right), \left(Y_1 + Y_2\sqrt{N}\right)$ of $F[R]$ by $\left(X_1 + X_2\sqrt{N}\right) \bullet \left(Y_1 + Y_2\sqrt{N}\right) = X_1Y_1 + NX_2Y_2 + (X_1Y_2 + Y_1X_2)\sqrt{N}\ mod\ p$ also.

2.1 Theorem

$(F[R], +, \bullet)$ is a field.

Proof:

The reader can use the above operations to check this.

This field will be called rhotrix quadratic extension. And since field extensions are regarded as vector spaces, the following could easily be observed:

2.2 Theorem

$F[R]$ is a vector space over $F_p[R]$.

Proof:

It is clear that $F_p[R]$ is a field which is properly contained in $F[R]$, since whenever $Y = 0$, then $XB_1 + XB_2 = X \epsilon F_p[R]$ and this will happen $p - times$. Therefore, $F[R]$, is a field extension of $F_p[R]$ and hence a vector space over it ■

2.3 Theorem

The basis for $F[R]$ is $B = \{B_1, B_2\}$

Proof:

Since every element of $F[R]$, is of the form $XB_1 + YB_2$, which equals

$$\left\langle \begin{array}{ccccccc} & & & 0_1 & & & \\ & & 0_2 & 0_3 & 0_4 & & \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \\ 0_\alpha & \cdots & \cdots & 0_\beta & \cdots & \cdots & 0_\pi \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \\ & & 0_{t-3} & 0_{t-2} & 0_{t-1} & & \\ & & & a_t & & & \end{array} \right\rangle$$

only when

$$X = Y = \left\langle \begin{array}{ccccccc} & & & 0_1 & & & \\ & & 0_2 & 0_3 & 0_4 & & \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \\ 0_{\alpha} & \cdots & \cdots & 0_{\beta} & \cdots & \cdots & 0_{\pi} \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \\ & & 0_{t-3} & 0_{t-2} & 0_{t-1} & & \\ & & & a_t & & & \end{array} \right\rangle$$

Hence, $XB_1 + XB_2$ is linearly independent and therefore, $\{B_1, B_2\}$ generated $F[R]$ that is $F[R] = \{XB_1 + XB_2 : X, Y \epsilon F_p[R]\}$ which means the basis for $F[R]$ is $B = \{B_1, B_2\}$■

2.4 Theorem

The characteristic of $F[R]$ is p.

Proof:

Pick an arbitrary element

$$\theta = \left\langle \begin{array}{ccccccc} & & & 0_1 & & & \\ & & 0_2 & 0_3 & 0_4 & & \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \\ 0_{\alpha} & \cdots & \cdots & a_{\beta} & \cdots & \cdots & 0_{\pi} \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \\ & & 0_{t-3} & 0_{t-2} & 0_{t-1} & & \\ & & & a_t & & & \end{array} \right\rangle \epsilon F[R]$$

then

$$p \times \theta = p\theta = \left\langle \begin{array}{ccccccc} & & & 0_1 & & & \\ & & 0_2 & 0_3 & 0_4 & & \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \\ 0_{\alpha} & \cdots & \cdots & a_{\beta} & \cdots & \cdots & 0_{\pi} \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \\ & & 0_{t-3} & 0_{t-2} & 0_{t-1} & & \\ & & & a_t & & & \end{array} \right\rangle$$

$$+ \left\langle \begin{array}{ccccccc} & & & 0_1 & & & \\ & & 0_2 & 0_3 & 0_4 & & \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \\ 0_{\alpha} & \cdots & \cdots & a_{\beta} & \cdots & \cdots & 0_{\pi} \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \\ & & 0_{t-3} & 0_{t-2} & 0_{t-1} & & \\ & & & a_t & & & \end{array} \right\rangle$$

$$+ \ \ldots$$

$$
+\left\langle \begin{array}{ccccccc} & & & 0_1 & & & \\ & & 0_2 & 0_3 & 0_4 & & \\ & \cdots & \ldots & \ldots & \ldots & \cdots & \\ 0_{\alpha} & \cdots & \ldots & a_{\beta} & \ldots & \cdots & 0_{\pi} \\ & \cdots & \ldots & \ldots & \ldots & \cdots & \\ & & 0_{t-3} & 0_{t-2} & 0_{t-1} & & \\ & & & a_t & & & \end{array} \right\rangle p - times
$$

$$
= \left\langle \begin{array}{ccccccc} & & & 0_1 & & & \\ & & 0_2 & 0_3 & 0_4 & & \\ & \cdots & \ldots & \ldots & \ldots & \cdots & \\ 0_{\alpha} & \cdots & \ldots & 0_{\beta} & \ldots & \cdots & 0_{\pi} \\ & \cdots & \ldots & \ldots & \ldots & \cdots & \\ & & 0_{t-3} & 0_{t-2} & 0_{t-1} & & \\ & & & a_t & & & \end{array} \right\rangle \blacksquare
$$

Since $|B| = 2$, we can also have the following:

2.5 Theorem

$|F[R]| = p^2$.

Proof:

This follows from the fact that field extensions can be regarded as vector spaces and $F[R]$ is of characteristic p with 2 as the dimension of its basis ■

Examples:

(1) Pick $F_3[R]$, if $t = 5$, then

Since $F_3[R] = \left\{ \left\langle \begin{array}{ccc} & 0 & \\ 0 & 0 & 0 \\ & 0 & \end{array} \right\rangle, \left\langle \begin{array}{ccc} & 0 & \\ 0 & 1 & 0 \\ & 0 & \end{array} \right\rangle, \left\langle \begin{array}{ccc} & 0 & \\ 0 & 2 & 0 \\ & 0 & \end{array} \right\rangle \right\}$

The only element that is not the square of any other element here is $\left\langle \begin{array}{ccc} & 0 & \\ 0 & 2 & 0 \\ & 0 & \end{array} \right\rangle$

That means the basis for this extension is

$$
B = \left\{ , \left\langle \begin{array}{ccc} & 0 & \\ 0 & 1 & 0 \\ & 0 & \end{array} \right\rangle, \sqrt{\left\langle \begin{array}{ccc} & 0 & \\ 0 & 2 & 0 \\ & 0 & \end{array} \right\rangle} \right\}
$$

and the set generated by the elements of B has nine elements as follows:

$$F[R] = \{f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8, f_9\}$$

such that

$$f_1 = \left\langle \begin{matrix} & 0 & \\ 0 & 0 & 0 \\ & 0 & \end{matrix} \right\rangle$$

$$f_2 = \left\langle \begin{matrix} & 0 & \\ 0 & 1 & 0 \\ & 0 & \end{matrix} \right\rangle$$

$$f_3 = \left\langle \begin{matrix} & 0 & \\ 0 & 2 & 0 \\ & 0 & \end{matrix} \right\rangle$$

$$f_4 = \sqrt{\left\langle \begin{matrix} & 0 & \\ 0 & 2 & 0 \\ & 0 & \end{matrix} \right\rangle}$$

$$f_5 = \left\langle \begin{matrix} & 0 & \\ 0 & 1 & 0 \\ & 0 & \end{matrix} \right\rangle + \sqrt{\left\langle \begin{matrix} & 0 & \\ 0 & 2 & 0 \\ & 0 & \end{matrix} \right\rangle}$$

$$f_6 = \left\langle \begin{matrix} & 0 & \\ 0 & 2 & 0 \\ & 0 & \end{matrix} \right\rangle + \sqrt{\left\langle \begin{matrix} & 0 & \\ 0 & 2 & 0 \\ & 0 & \end{matrix} \right\rangle}$$

$$f_7 = \left\langle \begin{matrix} & 0 & \\ 0 & 2 & 0 \\ & 0 & \end{matrix} \right\rangle \sqrt{\left\langle \begin{matrix} & 0 & \\ 0 & 2 & 0 \\ & 0 & \end{matrix} \right\rangle}$$

$$f_8 = \left\langle \begin{matrix} & 0 & \\ 0 & 1 & 0 \\ & 0 & \end{matrix} \right\rangle + \left\langle \begin{matrix} & 0 & \\ 0 & 2 & 0 \\ & 0 & \end{matrix} \right\rangle \sqrt{\left\langle \begin{matrix} & 0 & \\ 0 & 2 & 0 \\ & 0 & \end{matrix} \right\rangle}$$

$$f_9 = \left\langle \begin{matrix} & 0 & \\ 0 & 2 & 0 \\ & 0 & \end{matrix} \right\rangle + \left\langle \begin{matrix} & 0 & \\ 0 & 2 & 0 \\ & 0 & \end{matrix} \right\rangle \sqrt{\left\langle \begin{matrix} & 0 & \\ 0 & 2 & 0 \\ & 0 & \end{matrix} \right\rangle}$$

The set $F[R]$ together with (+) and (•) defined above is a field.

Note that this construction is only possible, when there exists at least one element which is not a perfect square in the base field, for instance, with $F_2[R] = \left\{ \left\langle \begin{array}{c} 0 \\ 0\ 0\ 0 \\ 0 \end{array} \right\rangle, \left\langle \begin{array}{c} 0 \\ 0\ 1\ 0 \\ 0 \end{array} \right\rangle \right\}$ this construction may not be possible.

(2) Pick $F_5[R]$, where $t = 5$ also,

Since $F_5[R] = \left\{ \left\langle \begin{array}{c} 0 \\ 0\ 0\ 0 \\ 0 \end{array} \right\rangle, \left\langle \begin{array}{c} 0 \\ 0\ 1\ 0 \\ 0 \end{array} \right\rangle, \left\langle \begin{array}{c} 0 \\ 0\ 2\ 0 \\ 0 \end{array} \right\rangle, \left\langle \begin{array}{c} 0 \\ 0\ 3\ 0 \\ 0 \end{array} \right\rangle, \left\langle \begin{array}{c} 0 \\ 0\ 4\ 0 \\ 0 \end{array} \right\rangle \right\}$

Here $\left\langle \begin{array}{c} 0 \\ 0\ 2\ 0 \\ 0 \end{array} \right\rangle$ and $\left\langle \begin{array}{c} 0 \\ 0\ 3\ 0 \\ 0 \end{array} \right\rangle$ are not squares of any elements, so any of them can be in the construction. If $\left\langle \begin{array}{c} 0 \\ 0\ 3\ 0 \\ 0 \end{array} \right\rangle$ is considered, then

$$B = \left\{ , \left\langle \begin{array}{c} 0 \\ 0\ 1\ 0 \\ 0 \end{array} \right\rangle, \sqrt{\left\langle \begin{array}{c} 0 \\ 0\ 3\ 0 \\ 0 \end{array} \right\rangle} \right\}$$

is the basis and the set generated by the elements of B will have 5^2 number of elements like this:

$$F[R] = \{f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8, f_9, f_{10}, f_{11}, f_{12}, f_{13}, f_{14}, f_{15}, f_{16}, f_{17}, f_{18}\ f_{19}, f_{20}, f_{21}, f_{22}, f_{23}, f_{24}, f_{25}\}$$

where

$$f_1 = \left\langle \begin{array}{c} 0 \\ 0\ 0\ 0 \\ 0 \end{array} \right\rangle$$

$$f_2 = \left\langle \begin{array}{c} 0 \\ 0\ 1\ 0 \\ 0 \end{array} \right\rangle$$

$$f_3 = \left\langle \begin{array}{c} 0 \\ 0\ 2\ 0 \\ 0 \end{array} \right\rangle$$

$$f_4 = \left\langle \begin{matrix} & 0 & \\ 0 & 3 & 0 \\ & 0 & \end{matrix} \right\rangle$$

$$f_5 = \left\langle \begin{matrix} & 0 & \\ 0 & 4 & 0 \\ & 0 & \end{matrix} \right\rangle$$

$$f_6 = \sqrt{\left\langle \begin{matrix} & 0 & \\ 0 & 3 & 0 \\ & 0 & \end{matrix} \right\rangle}$$

$$f_7 = \left\langle \begin{matrix} & 0 & \\ 0 & 2 & 0 \\ & 0 & \end{matrix} \right\rangle \sqrt{\left\langle \begin{matrix} & 0 & \\ 0 & 3 & 0 \\ & 0 & \end{matrix} \right\rangle}$$

$$f_8 = \left\langle \begin{matrix} & 0 & \\ 0 & 3 & 0 \\ & 0 & \end{matrix} \right\rangle \sqrt{\left\langle \begin{matrix} & 0 & \\ 0 & 3 & 0 \\ & 0 & \end{matrix} \right\rangle}$$

$$f_9 = \left\langle \begin{matrix} & 0 & \\ 0 & 4 & 0 \\ & 0 & \end{matrix} \right\rangle \sqrt{\left\langle \begin{matrix} & 0 & \\ 0 & 3 & 0 \\ & 0 & \end{matrix} \right\rangle}$$

$$f_{10} = \left\langle \begin{matrix} & 0 & \\ 0 & 1 & 0 \\ & 0 & \end{matrix} \right\rangle + \sqrt{\left\langle \begin{matrix} & 0 & \\ 0 & 3 & 0 \\ & 0 & \end{matrix} \right\rangle}$$

$$f_{11} = \left\langle \begin{matrix} & 0 & \\ 0 & 2 & 0 \\ & 0 & \end{matrix} \right\rangle + \sqrt{\left\langle \begin{matrix} & 0 & \\ 0 & 3 & 0 \\ & 0 & \end{matrix} \right\rangle}$$

$$f_{12} = \left\langle \begin{matrix} & 0 & \\ 0 & 3 & 0 \\ & 0 & \end{matrix} \right\rangle + \sqrt{\left\langle \begin{matrix} & 0 & \\ 0 & 3 & 0 \\ & 0 & \end{matrix} \right\rangle}$$

$$f_{13} = \left\langle \begin{matrix} & 0 & \\ 0 & 4 & 0 \\ & 0 & \end{matrix} \right\rangle + \sqrt{\left\langle \begin{matrix} & 0 & \\ 0 & 3 & 0 \\ & 0 & \end{matrix} \right\rangle}$$

$$f_{14} = \left\langle \begin{matrix} & 0 & \\ 0 & 1 & 0 \\ & 0 & \end{matrix} \right\rangle + \left\langle \begin{matrix} & 0 & \\ 0 & 2 & 0 \\ & 0 & \end{matrix} \right\rangle \sqrt{\left\langle \begin{matrix} & 0 & \\ 0 & 3 & 0 \\ & 0 & \end{matrix} \right\rangle}$$

$$f_{15} = \left\langle \begin{matrix} & 0 & \\ 0 & 2 & 0 \\ & 0 & \end{matrix} \right\rangle + \left\langle \begin{matrix} & 0 & \\ 0 & 2 & 0 \\ & 0 & \end{matrix} \right\rangle \sqrt{\left\langle \begin{matrix} & 0 & \\ 0 & 3 & 0 \\ & 0 & \end{matrix} \right\rangle}$$

$$f_{16} = \left\langle \begin{matrix} & 0 & \\ 0 & 3 & 0 \\ & 0 & \end{matrix} \right\rangle + \left\langle \begin{matrix} & 0 & \\ 0 & 2 & 0 \\ & 0 & \end{matrix} \right\rangle \sqrt{\left\langle \begin{matrix} & 0 & \\ 0 & 3 & 0 \\ & 0 & \end{matrix} \right\rangle}$$

$$f_{17} = \left\langle \begin{matrix} & 0 & \\ 0 & 4 & 0 \\ & 0 & \end{matrix} \right\rangle + \left\langle \begin{matrix} & 0 & \\ 0 & 2 & 0 \\ & 0 & \end{matrix} \right\rangle \sqrt{\left\langle \begin{matrix} & 0 & \\ 0 & 3 & 0 \\ & 0 & \end{matrix} \right\rangle}$$

$$f_{18} = \left\langle \begin{matrix} & 0 & \\ 0 & 1 & 0 \\ & 0 & \end{matrix} \right\rangle + \left\langle \begin{matrix} & 0 & \\ 0 & 3 & 0 \\ & 0 & \end{matrix} \right\rangle \sqrt{\left\langle \begin{matrix} & 0 & \\ 0 & 3 & 0 \\ & 0 & \end{matrix} \right\rangle}$$

$$f_{19} = \left\langle \begin{matrix} & 0 & \\ 0 & 2 & 0 \\ & 0 & \end{matrix} \right\rangle + \left\langle \begin{matrix} & 0 & \\ 0 & 3 & 0 \\ & 0 & \end{matrix} \right\rangle \sqrt{\left\langle \begin{matrix} & 0 & \\ 0 & 3 & 0 \\ & 0 & \end{matrix} \right\rangle}$$

$$f_{20} = \left\langle \begin{matrix} & 0 & \\ 0 & 3 & 0 \\ & 0 & \end{matrix} \right\rangle + \left\langle \begin{matrix} & 0 & \\ 0 & 3 & 0 \\ & 0 & \end{matrix} \right\rangle \sqrt{\left\langle \begin{matrix} & 0 & \\ 0 & 3 & 0 \\ & 0 & \end{matrix} \right\rangle}$$

$$f_{21} = \left\langle \begin{matrix} & 0 & \\ 0 & 4 & 0 \\ & 0 & \end{matrix} \right\rangle + \left\langle \begin{matrix} & 0 & \\ 0 & 3 & 0 \\ & 0 & \end{matrix} \right\rangle \sqrt{\left\langle \begin{matrix} & 0 & \\ 0 & 3 & 0 \\ & 0 & \end{matrix} \right\rangle}$$

$$f_{22} = \left\langle \begin{matrix} & 0 & \\ 0 & 1 & 0 \\ & 0 & \end{matrix} \right\rangle + \left\langle \begin{matrix} & 0 & \\ 0 & 4 & 0 \\ & 0 & \end{matrix} \right\rangle \sqrt{\left\langle \begin{matrix} & 0 & \\ 0 & 3 & 0 \\ & 0 & \end{matrix} \right\rangle}$$

$$f_{23} = \left\langle \begin{matrix} & 0 & \\ 0 & 2 & 0 \\ & 0 & \end{matrix} \right\rangle + \left\langle \begin{matrix} & 0 & \\ 0 & 4 & 0 \\ & 0 & \end{matrix} \right\rangle \sqrt{\left\langle \begin{matrix} & 0 & \\ 0 & 3 & 0 \\ & 0 & \end{matrix} \right\rangle}$$

$$f_{24} = \left\langle \begin{matrix} & 0 & \\ 0 & 3 & 0 \\ & 0 & \end{matrix} \right\rangle + \left\langle \begin{matrix} & 0 & \\ 0 & 4 & 0 \\ & 0 & \end{matrix} \right\rangle \sqrt{\left\langle \begin{matrix} & 0 & \\ 0 & 3 & 0 \\ & 0 & \end{matrix} \right\rangle}$$

$$f_{25} = \left\langle \begin{matrix} & 0 & \\ 0 & 4 & 0 \\ & 0 & \end{matrix} \right\rangle + \left\langle \begin{matrix} & 0 & \\ 0 & 4 & 0 \\ & 0 & \end{matrix} \right\rangle \sqrt{\left\langle \begin{matrix} & 0 & \\ 0 & 3 & 0 \\ & 0 & \end{matrix} \right\rangle}$$

The set $F[R]$ above together with (+) and (•) is a field.

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