ON SIMILARITY SOLUTION OF THE TEMPERATURE FIELD OF A REACTING POROUS MEDIUM WITH A VARIBALE THERMAL CONDUCTIVITY

A.J Omowaye

Department of Mathematical Sciences

Federal University of Technology, Akure Nigeria.

Abstract

The temperature field of a reacting porous medium with a variable thermal conductivity is investigated. The resulting partial differential equation obtained was reduced to ordinary differential equation with help of similarity variable. Further more, the non-linear ordinary differential equation obtained was solved numerically using shooting technique. Numerical results are introduced in graphical form for different values of the parameters involved in the problem. Finally, the existence and uniqueness of solution of the mode was checked and it was found that the model has a unique solution.

Key words: Similarity solution, porous media, thermal conductivity, reacting flow, numerical solution.

AMS Classification: 76805,78M10

1 Introduction

The study of transport phenomena in porous medium has attracted the attention of theorists and experimentalists in recent years, due to its scope in various fields of engineering and environmental sciences. In recent times ,porous media model are being applied for simulating more generalized situations such as flow through packed-bed, chemical reactors ,transpiration cooling, building thermal insulation ,petroleum reservoir, geothermal operations and ground-water hydrology(see [1]). Also ,majority of the studies on convection heat transfer in porous media are based on Darcy's law[2].

Many scientists have contributed to the literature of flows through porous media. Barenblatt and Vazquez[3] revisited the theory of filtration through a horizontal porous stratum under the usual conditions of gently sloping height profile. A model for flooding followed by natural outflow through the end –wall of the stratum was considered . The proposed model leads to a kind of free boundary problem for the Boussinesq equation(porous medium equation) the flux as well as the height of the fluid was prescribed in terms of self-similar solution via phase plane techniques. Ayeni[4] presented a phase plane analysis of a liquid front moving through a hot porous rock. The mechanism by which relative cold liquid vapourises as it invades a hot permeable rock and the high rise in the pressure of the system were broadly considered. Calugaru et al [5] proposed a new model for two viscous fluids in porous media ,this generalizes previous model by removing the classical condition of gravity equilibrium and by considering a non-stationary flow (the compressibility of the fluids and of the porous medium is not neglected).Instead of previous hypothesis ,a more general assumption about linear behavior of the vertical velocity along vertical coordinate is used. This generalized model has been already applied to describe the flow of rather compressible fluids, when an elastic perturbation is propagating in the domain much slower than a gravity perturbation .While Calugaru et al [6] deals with application of this model in realistic situation where it can be supposed that gravity perturbations are propagating much slower than elastic perturbations among these applications ,one can include the classical well-known case of ground water flow with free surface, but also more complex phenomena, as gravitational instability with finger growth. Makinde [7] examined the steady-state solutions of a strongly exothermic reaction of a viscous combustible material in a cannel filled a saturated porous medium under Arrhenius kinetics, neglecting reactant consumption ,he model the problem using Brinkman model and analytical solutions are constructed for the governing nonlinear boundary –value problem using perturbation technique. Ogunsola and Ayeni [8] examined a long horizontal porous stratum bounded from below by an impermeable horizontal bottom and on the side by a vertical plane. The mass conservation law and the momentum equation for porous media with variable permeability in a free boundary model were considered . The model leads to a porous medium equation with variable permeability. A similarity solution was obtained using power series method of the solution for the flow velocity expressed in terms of the hydraulic head of the fluid in the medium. Omowaye and Ayeni [9] reported the effect of permeability on the temperature field in a reacting porous medium, they assumed that the ratio of the product of the universal gas constant and the initial temperature to the activation energy is of order one. In particular they showed that maximum temperature rise decreases as the permeability increases. Phiri and Makinde [10] reported the effects of chemical reaction with heat and mass transfer on unsteady flow of an electrically conducting viscous fluid past an accelerating vertical porous plate in the presence of a transversely imposed external magnetic field and heat generation or absorption. The plate was embedded in a uniform Darcian porous medium in order to allow for possible fluid wall suction or blowing and has a variable wall temperature and concentration. The coupled partial differential equations describing the conservation of mass, momentum and energy were obtained and solved analytically. Scheudegger[11] discussed the phenomenous of inhibition in porous medium under certain conditions and obtained its similarity transformation . Meanwhile Olajuwon [12] investigated unsteady temperature field of a power-law fluid flow with variable thermal conductivity ,he shown among other result that the heat dissipation is essential while thermal conductivity varies with time .Wood [13] in his lecture series on porous media with emphasis on ground water flows ,expounded much on dispersion and diffusion in porous medium flows .He explained some types of reactions driven by fluid flows in porous rocks and affirmed that phase changes occur as consequences of background temperature variations due to fluid flows. He discussed the phenomena (advection and diffusive heat flux)in porous media, mentioning the basic laws of thermodynamics in such media.

The primary aim of this paper is to examined the effect of certain fluid parameters on the temperature field of a reacting fluid in porous medium when the thermal conductivity of the fluid is not constant also the existence and uniqueness theorem will be establish for the model obtained.

2 Mathematical formulation

Consider an unsteady flow of a reacting fluid through a porous medium ,the medium is assumed to be homogenous and isotropic and characterized by constant porosity. The thermal conductivity varies as a function of temperature i.e $\lambda = \lambda(T)$ .Also, the Darcy law was consider as given by Henry Darcy (1856) We shall restrict consideration to one dimensional geometries for convenience and brevity. It is to be noted however that similar problems can be naturally posed in multi-dimensional geometries. Based on these assumptions above the governing equations are given as follows

$$\frac{\mu}{k} u + grad \quad p = 0 \qquad \text{(Darcy Law)} \qquad (1)$$

$$\rho c_p \left(\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial y}\right) = \frac{\partial}{\partial y}\left(\lambda(T)\frac{\partial T}{\partial y}\right) + \frac{Q_0 (T - T_0)}{t} \qquad (2)$$

The physical boundary conditions for the problem are

$$T(0,t) = T_0 \qquad t>0 \qquad T_1 > T_0 \qquad (3)$$
$$T(1,t) = T_1$$

where

μ - dynamic viscosity

u - velocity

p - pressure

ρ - density

C_p - specific heat capacity at constant pressure

T -temperature

λ(T)- variable thermal conductivity

Q_0 – heat release per unit mass

t - time

y - space variable

T_0 - initial temperature

T_1- Final temperature

2 Method of solution

We consider the thermal conductivity to be of the form

$$\lambda(T) = (a + bT)^n \qquad (4)$$

where a and b are constants while n is an integer. For simplicity and for the purpose of this paper we shall take our n=1 .Also, let

$$p = \frac{y}{t^{1/2}} \qquad (5)$$

then equations (1) and (2) becomes

$$\frac{\partial T}{\partial t} - \frac{k}{\mu\, t^{1/2}}\frac{\partial T}{\partial y} = \frac{1}{\rho c_p}(a + bT)\frac{\partial^2 T}{\partial y^2} + \frac{b}{\rho c_p}\left(\frac{\partial T}{\partial y}\right)^2 + \frac{Q_0\,(T - T_0)}{\rho c_p} \qquad (6)$$

We seek a self-similar solution of the form

$$\eta = \frac{Ay}{t^{\alpha}} \qquad g(\eta) = T(y,t) \qquad (7)$$

Putting equation (7) into equation (8) we obtain

$$\frac{A^2}{\rho c_p}(a + bg)\,g'' + \frac{bA^2}{\rho c_p}(g')^2 + \left(\alpha\eta + \frac{kA}{\mu}\right)g' + \frac{Q_0}{\rho C_P}\,g = 0 \qquad (8)$$

Subject to

$$g(0) = T_0 \qquad g(1) = T_1 \qquad (9)$$

$$\xi A^2\,(a + bg)\,g'' + \xi b A^2 (g')^2 + (\alpha\eta + DA)g' + \xi\, Q_0\, g = 0 \qquad (10)$$

where $D = \frac{K}{\mu}$ $\qquad \xi = \frac{1}{\rho c_p}$

3. Existence and Uniqueness of Solution

In this section we shall examine the existence and Uniqueness of problem(10) subject to (9).

Theorem 1: For $0 \le \eta \le 1$; $\xi, a, b, A, D, Q_0, \alpha > 0$ are constants. Then problem

(10) satisfying (9) has a unique solution.

Proof:

We transform problem (10) into system of first order differential equations as follows

Let $g_1 = \eta \ , \ g_2 = g \ , \ g_3 = g'$

$$\begin{pmatrix} g_1' \\ g_2' \\ g_3' \end{pmatrix} = \begin{pmatrix} 1 \\ g_3 \\ \dfrac{-\xi\, b\, A^2 (g_3)^2 - (\alpha\, g_1 + DA) g_3 - Q_0 \xi\, g_2}{\xi\, A^2 (a + b g_2)} \end{pmatrix} \qquad (11)$$

together with the initial conditions

$$\begin{pmatrix} g_1(0) \\ g_2(0) \\ g_3(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \gamma \end{pmatrix} \qquad (12)$$

where γ is guess such that the boundary condition is satisfied.

Now ,we define

$$f_1(g_1, g_2, g_3) = 1$$
$$f_2(g_1, g_2, g_3) = g_3$$
$$f_3(g_1, g_2, g_3) = \frac{-\xi\, b\, A^2 (g_3)^2 - (\alpha g_1 + DA) g_3 - Q_0 \xi\, g_2}{\xi\, A^2 (a + b g_2)}$$

Theorem2:

For

$0 \le g_1 \le 1,\ 0 \le g_2 \le M,\ \gamma \le g_3 \le \gamma^*, a, b, A, D, Q_0, \gamma, \gamma^*, \alpha, \xi, M > 0$ the function $f_i\,(i=1,2,3)$ are Lipschitz continuous.

Proof:

Let

$$\left|\frac{\partial f_1}{\partial g_j}\right| = 0, i=1,2,3, \quad \left|\frac{\partial f_2}{\partial g_1}\right| = 0, \left|\frac{\partial f_2}{\partial g_2}\right| = 0, \left|\frac{\partial f_2}{\partial g_3}\right| = 1, \left|\frac{\partial f_3}{\partial g_1}\right| = \left|\frac{-\alpha g_3}{\xi A^2(a+b g_2)}\right| = N_2 < \infty$$

$$\left|\frac{\partial f_3}{\partial g_2}\right| = \left|\frac{-Q_0\xi}{A^2(a+b g_2)^2}\right| = N_3 < \infty, \quad \left|\frac{\partial f_3}{\partial g_3}\right| = \left|\frac{-2\xi b A^2 g_3 - (\alpha g_1 + DA)}{\eta A^2(a+bg_2)}\right| = N_4 < \infty$$

The partial derivatives $\frac{\partial f_i}{\partial g_j}$, $i, j = 1,2,3$ are bounded since there exist Lipschitz constant k> 0, such that $\left|\frac{\partial f_i}{\partial g_j}\right| \le k \quad i, j = 1,2,3$

.Hence, $f_i(g_1, g_2, g_3)$ i=1,2,3 are Lipschitz continuous and so (11) satisfying (12) is Lipschitz continuous .

Proof of theorem 1 :

The existence of Lipschitz constant in the proof of theorem 2 implies the existence of unique solution of problem (10) satisfying (9) and consequently the existence of unique solution. This completes the proof.

4. Numerical solution

Equation (10) subject to equation (9) does not have close form solution. We therefore employed the numerical method called shooting technique. The basic idea of shooting method for solving boundary value ordinary differential equation is to try to find approximate initial condition for which the computed solution “hits the target” so that the boundary conditions at other points are satisfied. In other words, the boundary value problem (10) subject to (9) is transform to system of initial value problem (11) subject to (12) where γ in (12) is guessed such that g (1)=2 is satisfied. The numerical result is presented in figures 1-4 below.(see [14] there in)

5. Numerical results and discussion

The use of similarity variable (7) transforms the porous media equation with variable thermal conductivity from partial differential equation to an ordinary differential equation. The polynomial dependence of the thermal conductivity on temperature results in decomposing the term in the energy equation into two terms Also, we monitor the effect of variable thermal conductivity, porous coefficient and other parameters in the medium as described by the model (10) using shooting technique. For illustration purposes the parameter values in table 1below are used.

Table 1:Set of parameters value

a=b	Q	D	ξ	α=A
2.0	2.0	0.25	0.5	1
2.5	3.0	0.45	0.9	1
3.0	4.0	0.65	1.3	1

The effect of different parameters entering the problem is illustrated graphically in figures (1-4).Fig 1 shows the graph of $g(\eta)$ against η for various values of a=b .It is found that the temperature increases as the value of a=b decreases .Fig 2 illustrate the plot of $g(\eta)$ against η for various values of Q_0. ,it shows the temperature of the medium is not stable. Fig 3 shows the graph of $g(\eta)$ against η for various values of D, The plot shows that temperature of the medium is unstable with various values of D. Fig 4 shows the plot of $g(\eta)$ against η for various values of ξ .The plot shows that the temperature of the medium decreases as the value of ξ increases

6 Conclusion

In this paper, the temperature field of a reacting porous medium with a variable thermal conductivity is considered .The criteria for existence of unique solution is established .Solution to the model was obtained via shooting technique. Finally, the results of this study will serve as baseline information to the hydrologist.

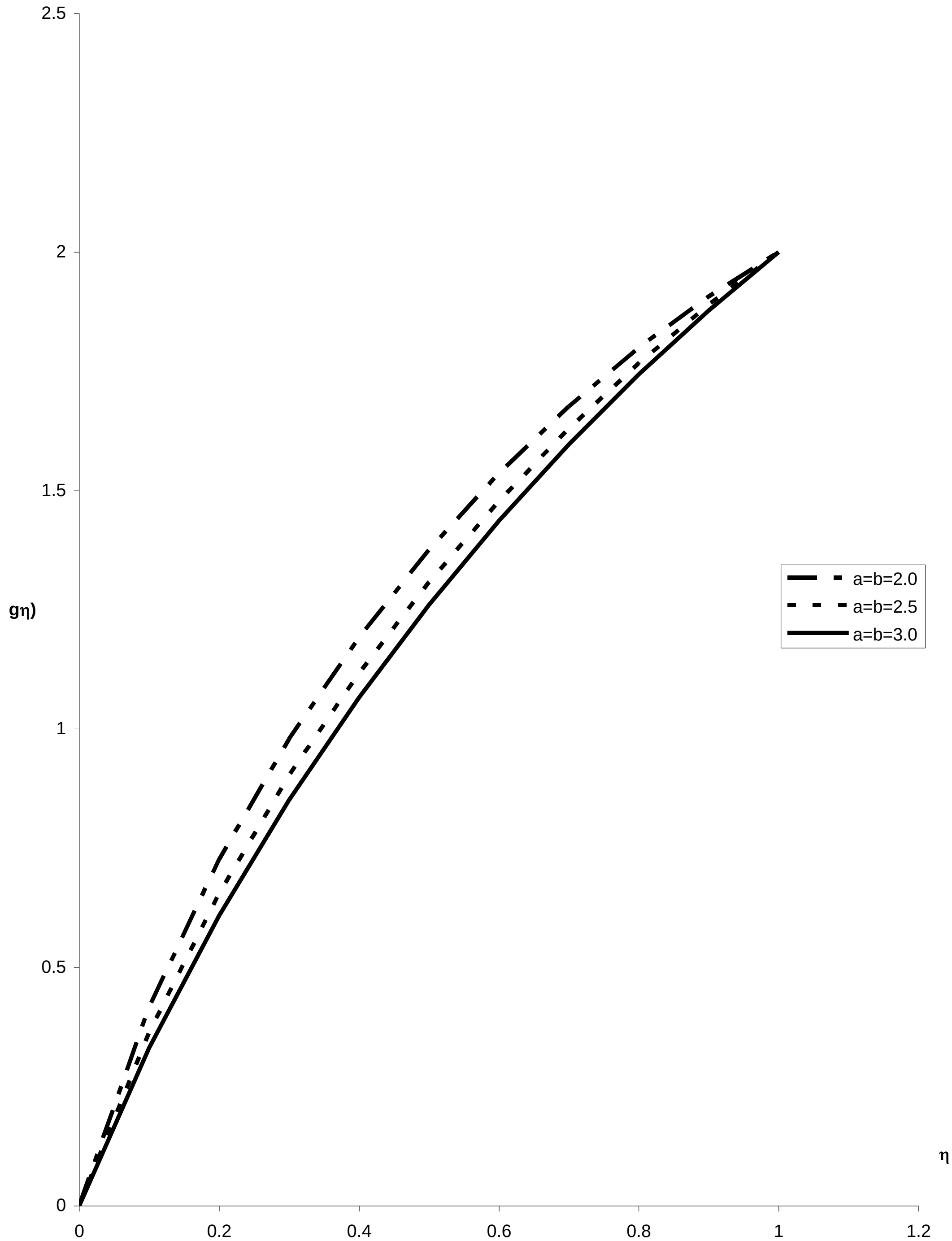

Fig 1:The graph of temperature profile for fixed value of Q=2.0,D=0.25,A=1,α=1,ξ=0.5 and for various a=b

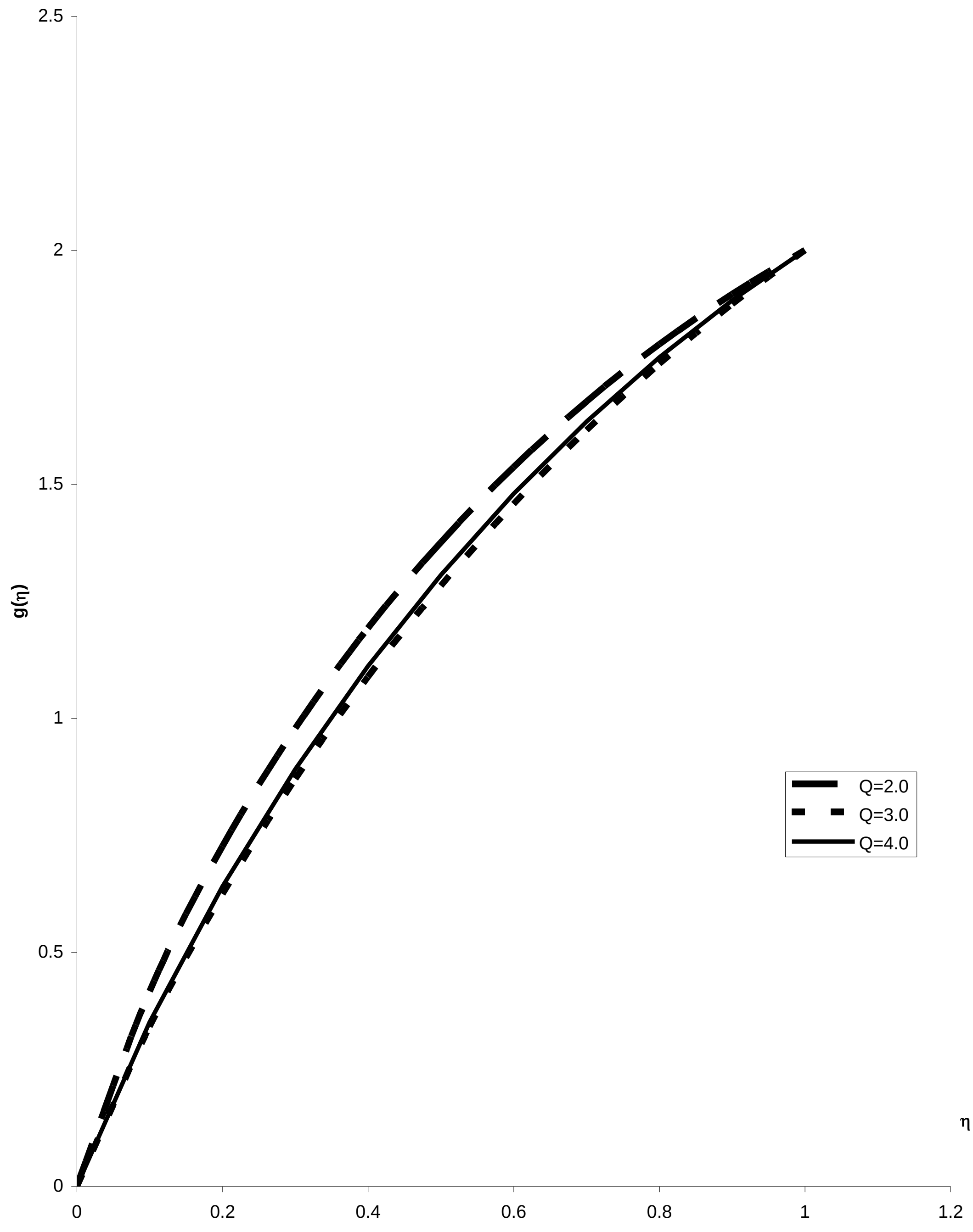

fIG 2:The graph of temperature profile for fixed value of A=1,D=0.25,a=b=2.0,α=1,ξ=0.5,for various Q

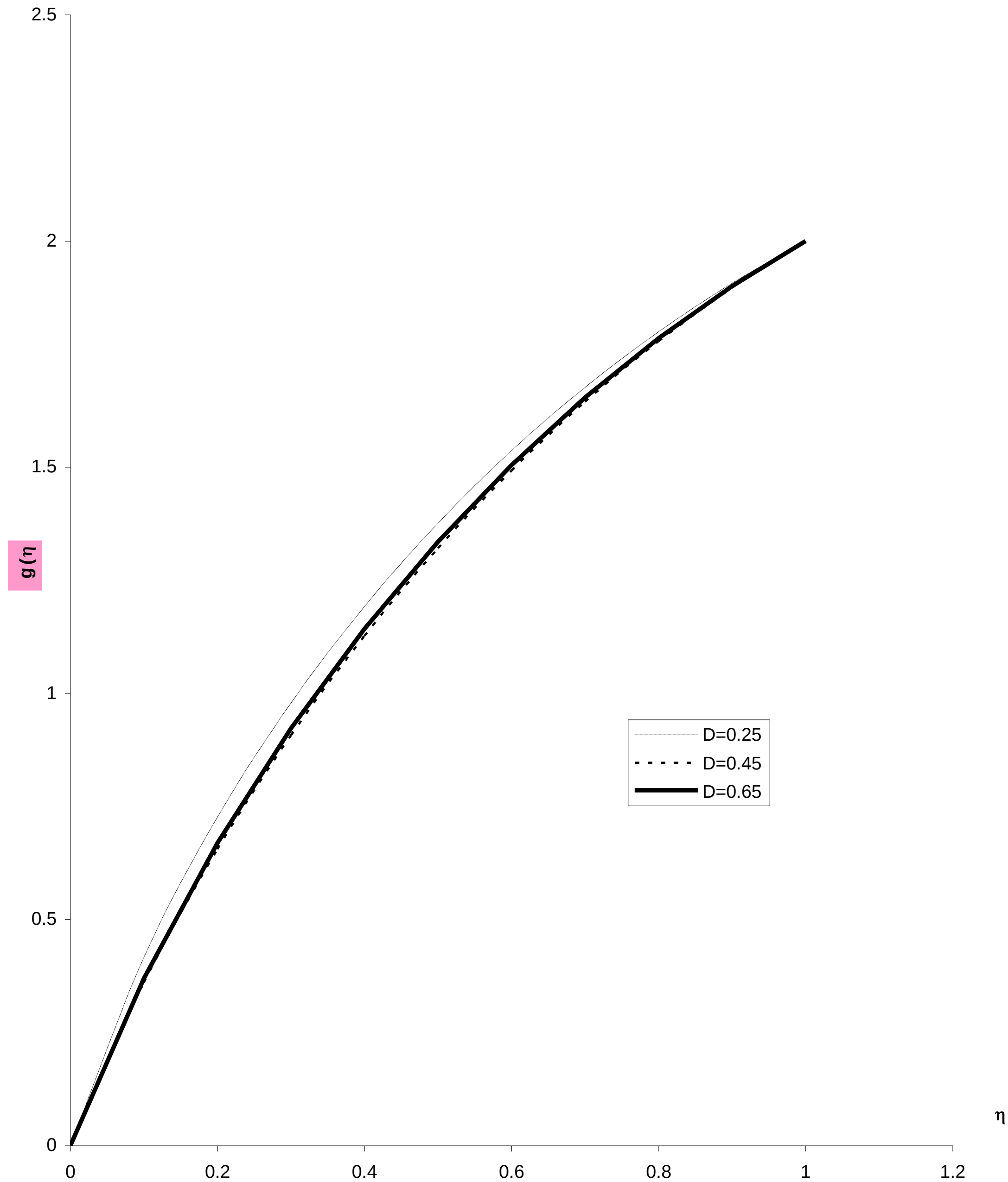

Fig 3:The graph of temperature profile for fixed value of A=1,a=b=2,α=1,ξ=0.5,Q=2.0,for various value of D

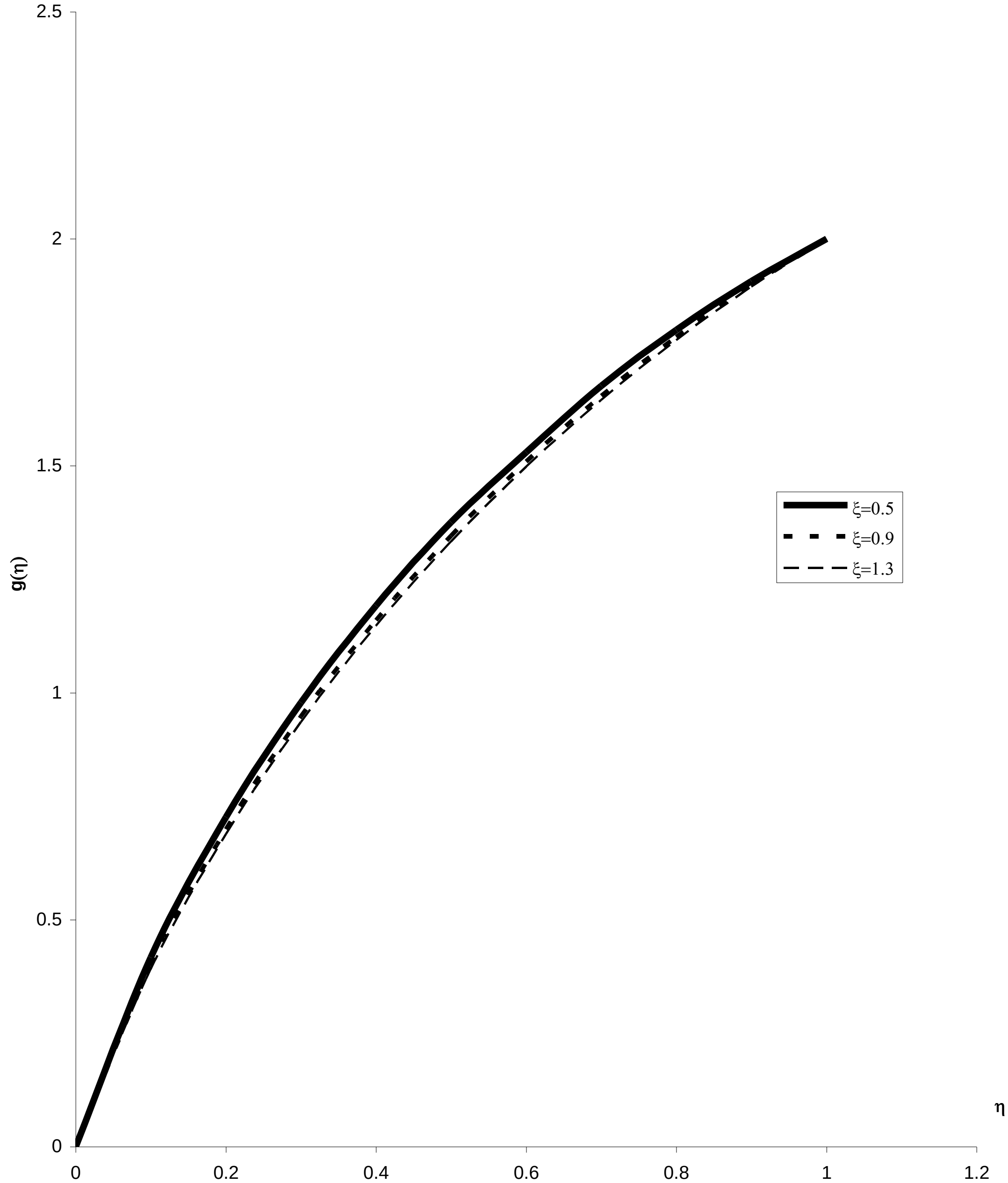

Fig 4:The graph of temperature profile for fixed value of A=1,D=0.25,a=b=2,Q=2.0,α=1 and for various ξ

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