# OPTIONAL STOPPING THEOREM AS AN INDISPENSABLE TOOL IN THE DETERMINATION OF RUIN PROBABILITY AND EXPECTED DURATION OF A GAME.

**BY**

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**ABSTRACT**

*This paper examines optional stopping theorem and its areas of applicability. It also examines two approaches in the determination of ruin probability and expected duration of a coin flipping game. First, an indirect approach based on the use of optional stopping theorem and second, a direct approach based on the use of Wald's first two identities. Having compared the two results, it was observed that both indirect and direct approach yield the same result for both ruin probability and expected duration of the game.*

Keywords: Optional Stopping Theorem, Coin flipping Game, Wald's first two Identities, Ruin Probability.

## 1.0 INTRODUCTION

Optional Stopping Theorem (OST) is a theorem which falls under the study of a discrete time martingales. However, a discrete time martingales falls under the study of a discrete time stochastic processes (i.e., a sequence of random variables $\{X_n, n \geq 0\}$. For OST to be applied to any given problem being defined by a given function, the function being defined must first acquire a martingale property. It will be wise enough for the readers to be provided with a little knowledge of martingales.

Martingale is an English word for *martegal* (French dialect word meaning inhabitant of Martigues, a village in France). The oldest meaning of martingale seems to be a piece of tack used on horses to control head carriage. Originally, a martingale referred to a class of betting strategy popular in the $18^{th}$ century France. The simplest of these strategies was designed for a coin flipping game. That is, a game in which the gambler wins his stake if the coin comes up 'heads' and loses if the coin comes up 'tails'. This strategy had the gambler (those who play games for bets) to double his bet after every loss.

Since the gambler with infinite wealth is guaranteed to eventually flip head, the martingale betting strategy was seen as a sure thing for those who practiced it.

Unfortunately, none of those practitioners' possessed infinite wealth and the exponential growth of bets would quickly bankrupt those foolish enough to use martingale after even a moderately long run of bad luck [1].

Another intuition about gambling as stated by Karlin and Taylor [2] is that a gambler cannot turn a fair game into an advantageous one by periodically deciding to double the bet or by cleverly choosing the time to quit playing. This intuition invariably led to OST.

There are various applications of martingales. For example Ugbebor and Ganiyu [3] applied the martingale model to the NGN/USD exchange rate. OST has many applications. For example, it was applied in risk theory by Gerber and Shiu [4-7]. The OST can also be applied to prove the impossibility of successful betting strategy of a gambler with a finite lifetime and a house limit on bet.

This paper examines a direct approach to the determination of ruin probability and expected duration of a game using Wald's first two identities. Also, the same type of problem was examined with an indirect approach based on the use of conditional expectation, Martingale properties, as well as OST. The two results are then compared. It was found that they are the same. It was noted that OST was telling us that even with a well chosen strategy for stopping a game, under some reasonable hypotheses a martingale is still a fair game.

## 2.0 PRELIMINARIES

### 2.1 Conditional Property of expectation

**Definition 2.1** Consider discrete random variable $X$ and $Y$. Let $S_X = \{x | P[X = x] > 0\}$. The conditional expectation of $Y$ given that $X = x$ has occurred, where $x \in S_X$ is defined by

$$E[Y|X](x) = E[Y|X = x] = \sum_y yP\{Y = y|X = x\} \tag{2.1}$$

**Theorem 2.1**

Let Y be independent of $X$ and $E[Y|X] = \psi(Y)$. Then, the function $\psi(X)$ satisfies $E[\psi(X)] = E[Y]$. **(See** reference no **[8]** for the proof**).**

**Definition 2.2** Consider random variables $X_1,...,X_n$. Denote by $\mathscr{F}_n$ the $\sigma-$**algebra** (i.e. collection of events) generated by these random variables which satisfies the properties (i) $\Omega \in \mathscr{F}_n$ (ii) $A \in \mathscr{F}_n \Rightarrow A^c \in \mathscr{F}_n$ and (iii) $A_1,...,A_k \in \mathscr{F}_n \Rightarrow \bigcup_{k=1}^{\infty} A_k \in \mathscr{F}_n$, then

$\mathscr{F}_n = \sigma\left(X_1,...,X_n\right), n \geq 0$.

**Definition 2.3** Let $X_1,...X_n$ be random variables. Also, let $\mathscr{F}_n, n \geq 0$ be the $\sigma-$algebra generated by the random variables. The random variable $Y$ is $\mathscr{F}_n$ **-measurable** if it depends on $X_1,...,X_n$, i.e. $\exists$ a function such that

$Y = f\left(X_1,...,X_n\right)$

**Theorem 2.2 (Linearity property of conditional expectation**

Let $Y, U$ and $V$ be discrete random variables. If the scalars $a, b \in \mathbb{R}$, then

$$E[\left(aU + bV\right)|Y] = aE\left(U|Y\right) + bE\left(V|Y\right) \tag{2.2}$$

**(See [8]** for the proof**).**

**Definition 2.4 A stochastic process** $\left\{X_n, n \geq 0\right\}$ is said to be a *martingale* with respect to a process $\left(Y_n, n \geq 0\right)$, if for all $n \geq 0$,

$$E\left[X_n\right] < \infty \text{ and } E\left[X_{n+1}|Y_0,...,Y_n\right] = X_n \tag{2.3}$$

**Remark 2.1** It should be noted that, by conditional expectation property which states that $E[g\left(X\right)|Y = y]$ is a function of $y$ for each $g$. For if we have $E\left|g\left(X\right)\right| < \infty$, $X_n$ is a function of $Y_1,...,Y_n$ determines the value of $X_n$. Also by the law of total probability for expectations,

$E\left(X_{n+1}\right) = E\left\{E\left[X_{n+1}|Y_0, Y_1,...,Y_n\right]\right\} = E\left(X_n\right)$

And thus by induction,

$$E\left(X_n\right) = E\left(X_0\right) \ \forall \ n \tag{2.4}$$

It is useful to think of $Y_0,...,Y_n$ as information or history up to stage $n$.

**Remark 2.1** If a game is a martingale, sub-martingale and super-martingale it is said to be fair, favourable and and unfavourable respectively.

**Example 2.1 Martingale, Sub-martingale and Super-martingale**

Consider a gambler who wins ₦100 when a coin comes up heads and loses ₦100 when the coin turns up tails. Let $Y_n = 1$ be the event that heads comes up and $Y_n = -1$ be the event that tails comes up. Suppose now that the coin may be biased such that $P(Y_n = 1) = p$, that is probability that the heads come up is $p$, and $P(Y_n = -1) = q = 1 - p$ i.e. the probability that the tails comes up is $q$.

(a) If $p = \frac{1}{2}$, then $P(Y_n = 1) = \frac{1}{2}$ and $P(Y_n = -1) = \frac{1}{2}$, the gambler's fortune over time is a ***martingale.***

(b) If $p < \frac{1}{2}$, e.g. $p = \frac{1}{4}$, then $P(Y_n = 1) = \frac{1}{4}$ and $P(Y_n = -1) = \frac{3}{4}$. This implies that $q > p$, then the gambler loses money on the average and the fortune over time is a ***sub-martingale.***

(c) If $p > \frac{1}{2}$, e.g. $p = \frac{3}{4}$, then If $P(Y_n = 1) = \frac{3}{4}$ and $P(Y_n = -1) = \frac{1}{4}$. This implies that $p > q$, then that gambler wins money on the average and the fortune overtime is a super-martingale.

**Theorem 2.3** If a gambler has equal chance of winning or losing a game and his betting strategy depends on the past history of the game, then the game is "fair" in the sense that the expected value of the next observation given the past history, is equal to the last observation. For the proof of this result **(see [8])**.

**Proposition 2.1**. Let $\{X_n\}$ be a martingale with respect to (w.r.t.) $\{\mathscr{F}_n\}$ or w.r.t. $\{Y_n\}$. Then

(i) $E\left[X_{n+k} | \mathscr{F}_n\right] = X_n$, for every $k$.

(ii) $E\left[X_n\right] = E\left[X_0\right]$, $\forall\ n$

For the proof of the result see (**see [8])**.

**Lemma 2.1**

Let $\{Y_n, n \geq 1\}$ be a sequence of independent and identically distributed random variables with mean zero and finite variance $\sigma^2$. Let $X_n = Y_1 + .. + Y_n$. Also, let $M_n := X_n^2 - n\sigma^2$. Then,

$\{M_n, n \geq 0\}$ is a martingale w.r.t. $\{X_n, n \geq 0\}$

**Proof:**

First note that $M_{n+1} = X_{n+1}^2 - (n+1)\sigma^2$

$\therefore M_{n+1} = (X_n + Y_{n+1})^2 - (n+1)\sigma^2$

$= X_n^2 + 2X_nY_{n+1} + Y_{n+1}^2 - n\sigma^2 - \sigma^2$

Taking the expected value and conditioned it on $Y_0,...,Y_n$,

$\therefore E\left[M_{n+1} | Y_0,...,Y_n\right] = E\left[(X_n^2 - 2X_nY_{n+1} + Y_{n+1}^2 - n\sigma^2 - \sigma^2) | Y_0,...,Y_n\right]$

$= E\left[X_n^2 | Y_0,...,Y_n\right] - 2E\left[(X_nY_{n+1} | Y_0,...,Y_n\right] + E\left[Y_{n+1}^2 | Y_0,...,Y_n\right] - n\sigma^2 - \sigma^2$ ( by Theorem 2.2)

$= E\left[X_n | Y_0,...,Y_n\right]\left[E\left[X_n | Y_0,...,Y_n\right] - 2E\left[X_n | Y_0,...,Y_n\right]\left[E\left[Y_{n+1} | Y_0,...,Y_n\right]\right]\right]$

$+\left(\left[E\left(Y_{n+1} | Y_0,...,Y_n\right)\right]^2 + \sigma^2\right) - n\sigma^2 - \sigma^2$, since $Y_n$ are independent.

$= X_n^2 - 2X_nE(Y_{n+1}) + \left[E(Y_{n+1})\right]^2 + \sigma^2 - n\sigma^2 - \sigma^2$

$= X_n^2 - 0 + 0^2 - n\sigma^2$, [since $E\left[Y_n\right] = 0 \Rightarrow E\left[Y_{n+1}\right] = 0$]

$= X_n^2 - n\sigma^2 = M_n$

$\{M_n, n \geq 0\}$ is a martingale w.r.t. $\{X_n, n \geq 0\}$.

**Remark 2.2** In general, it should be noted that if we have assumed a unit variance in our study, i.e. if we let $\sigma^2 = 1$, we can define $M_n := X_n^2 - n$. The same result follows using $\sigma^2 = 1$.

**Definition 2.5** Let $\{X_n\}, n \geq 0$ be a discrete time stochastic process, and $\mathscr{F}_n$ be the $\sigma$-algebra generated by $\{X_0,...,X_n\}$. A mapping $t : \Omega \to \{0,1,...,\infty\}$ is called a ***stopping time*** w.r.t. $\{X_n\}$ (or w.r.t. $\{\mathscr{F}_n\}$) if the event $\{t = n\}$ is completely determined by $\{X_0,....,X_n\}$ (or is a set in $\mathscr{F}_n$).

**Remark 2.3** By **"determined"**, we mean that the indicator function of the event $\{t = n\}$ can be written as a function of $X_0,...,X_n$, so that we can decide on whether or not $t = n$ from knowledge of the process $\{X_n\}$ only up to time $n$. We signify this by writing

$$I_{\{t=n\}} = I_{\{t=n}(X_0,...X_n) = \begin{cases} 1 & if \quad t=n \\ 0 & if \quad t \neq n \end{cases}$$

We often omit mentioning of $X_n$ and say that $t$ is a stopping time.

**3.0 OPTIONAL STOPPING THEOREM [OST]**

The OST as stated and proved by Kannan [1] can be stated as follows.

Let $\{X_n\}, n \geq 0$ be a martingale and $t$ a stopping time. If

(i) $P(t < \infty) = 1$

(ii) $E\left|X_t\right| < \infty$

(ii) $\lim_{x \to \infty} E\left[X_n I_{\{t>n\}}\right] = 0$

Then,

$E\left[X_t\right] = E\left[X_0\right]$.

For the proof**, (see [8]).**

**3.1 Applications of Optional Stopping Theorem**

Our focus in this paper is to compare the results which could be obtained by the application of optional stopping theorem (direct approach) and Wald's first two identities in the determination of ruin probability and expected duration of a coin flipping game. However, we are interested in providing two other applications as follows.

- 'Optional Stopping Theorem' asserts that a gambler can not improve his expected gain (fortune) having being given a (finite life time) stopping time (which gives conditions (i) and (ii) of the above theorem and a house limit on bets $\left\{\lim_{x \to \infty} E\left[X_n I_{\{t>n\}}\right] = 0\right\}$ (which gives condition (iii) of the theorem) i.e. the expected fortune of the gambler with an infinite wealth is zero. Therefore, OST can be used to prove the impossibility of successful betting strategy.

- Consider a random walk that starts at $0$ and stops if it reaches $-m$ or $+m$, and use the $Y_n = X_n^2 - n$ martingale from the result shown in Lemma 2.1 for the case when $\sigma = 1$.

If $t$ is the time at which it first reaches $\pm m$, then $E\left[Y_1\right] = E\left[Y_t\right] = m^2 - E\left[t\right]$. Since the walk starts from $0$, then $Y_0 = 0 \Rightarrow E\left[Y_0\right] = 0$. Hence, $E\left[Y_t\right] = E\left[Y_0\right] = m^2 - E\left[t\right] = 0$. This gives $E\left[t\right] = m^2$.

## 4.0 DIRECT AND INDIRECT APPROACH TO THE DETERMINATION OF RUIN PROBABILITY AND EXPECT DURATION OF A GAME.

### 4.1 A Direct Approach Based on the use of Wald's First Two Identities to determine ruin probability and expected duration of a game.

**Proposition 4.1 (Wald's First Identity).**

Let the jump variate $Y_t, t \geq 1$ have (common) mean $\mu$, $\left(E\left[Y_k\right] = \mu, \forall k\right)$,

and $P\left\{Y_t = 0\right\} = r < 1$. Then

$$E\left[X_t\right] = a + \mu E\left[t\right],$$

where $a = X_0 = Y_0$.

(**see [1] for the proof**])

**Proposition 4.2 (Wald's Second Identity)**

Let $\sigma^2$ be a unit variance of the jump variate $Y_t, t \geq 1$. Then

$$\text{Variance}\left(X_t\right) = \sigma^2 E\left(t\right).$$

(**see [1] for the proof**])

#### 4.1.1 Statement of the problem

In the problem below, we are interested in using Wald's first two identities to determine gambler's ruin probability and expected duration of a coin flipping game.

Two gamblers Dick and Harry play the following game. Dick repeatedly tosses a fair coin. After each toss that comes up Heads, Harry pays Dick ₦100. After each toss that comes up Tails, Dick pays Harry ₦100. The game continues until either one or the other gambler runs out of money. Let ₦$a$ and ₦ $b$ be the initial fortune of Dick and Harry respectively and $\{X_n, n \geq 0\}$ be a random walks corresponding to Dick's cumulative fortune. Let $P\left\{Y_n = 0\right\} = r < 1$. If $p = q$, where $p, q$ are probability of success and failure respectively. Then

(i) $P\left\{X_t = a + b\right\} = P\,\{\text{Harry is ruined}\} = \dfrac{a}{a+b}$ and dick has all the cash.

(ii) The expected duration of the game, $E\left(t\right) = ab$. C.f. Kannan (1997).

**Proof:**

Let $p$ and $q$ be probability of success and failure respectively. Let $Y_k = 1$ and $Y_k = -1$ be the event that Dick has all the cash and Harry is ruined, where $k \geq 0$. Then

$P\{Y_k = 1\} = p$ and $P\{Y_k = -1\} = q$.

$\therefore\ E[Y_k] = 1.p + (-1)q = p - q$

Since $p = q$, then, $E[Y_k] = p - q = 0 \Rightarrow \mu = E[Y_k] = 0$ (4.1)

Consequently, from Wald's first identity, $E[X_t] = a + \mu E(t)$

We therefore have

$E[X_t] = a$, (by 4.1 where $\mu = 0$) (4.2)

It should be noted that

$P(X_t = a+b) = P$ (Harry is ruined) and $P(X_t = 0) = P$ (Dick is ruined) (4.3)

$\therefore E(X_t) = (a+b).P(X_t = a+b) + 0.P(X_t = 0)$

$\Rightarrow E(X_t) = (a+b)P(X_t = a+b)$ (4.4)

Substitute $a = E(X_t)$ of equation (4.1) into equation (4.4), we have

$a = E(X_t) = (a+b)P(X_t = a+b) \ \Rightarrow P(X_t = a+b) = \dfrac{a}{a+b}$.

$\therefore\ P$ (Harry is ruined) $= \dfrac{a}{a+b}$

**Remark 4.2** It should be noted that $P$ (Dick is ruined) $= \dfrac{a}{a+b}$ can also be determined in a similar manner. In this case, Harry has all the cash.

(ii) Now if $p = q$

$Var(X_t) = E(X_t^2) - [E(X_t)]^2$

$\qquad = E(X_t^2) - a^2$, by (4.2)

$Var(X_t) = (a+b)^2 \cdot P(X_t = a+b) + 0^2 \cdot P(X_t = 0) - a^2$, by equation (4.3)

$\qquad = (a+b)^2 \cdot P(X_t = a+b) - a^2$ (4.5)

Substitute $P(X_t = a+b) = \dfrac{a}{a+b}$ in equation (4.5) above, we have

$$Var\left(X_t\right)=\left(a+b\right)^2\cdot\frac{a}{a+b}-a^2=ab$$

Hence from Wald's second identity

$$E\left(t\right)=Var\left(X_t\right)=ab.$$

Therefore, the expected duration of the game is $E\left(t\right)=ab$.

**4.2 An Indirect Approach Based on the use of Optional Stopping Theorem to determine ruin probability and expected duration of a game.**

**4.2.1 Statement of the problem**

Consider a gambler playing a coin flipping game. Let $Y_1,...,Y_n$ be a sequence of independent and identically distributed Bernoulli random variable with

$$P\left[Y_k=+1\right]=\frac{1}{2}=P\left[y_K=-1\right]\text{ for all } k.$$

Let $X_0$ be a given initial State (capital) and let $X_0+\sum_{k=1}^{n}Y_k$ be the accumulated fortune of the gambler at time $n$. Then $\{X_n\}$ is a martingale with respect to $\{Y_n\}$. Let

$$t:=\min\left\{n:X_n=-a\quad or\qquad X_n=b\right\}$$

Assume that the gambler is ruined if $X_n=-a$ and victorious if $X_n=b$. Furthermore, if $X_n$ reaches $-a$ before reaching $b$, then

(i) the ruin probability of the gambler is $r=\frac{b-X_0}{\left(a+b\right)}$ and

(ii) the expected duration of the game is $E\left(t\right)=\left(X_0+a\right)\left(b-X_0\right)$.

**Proof:**

First, we show that if

$$X_n=X_0+\sum_{k=1}^{n}Y_k \tag{4.6}$$

$\{X_n\}$ is a martingale.

Second, we should note that the gambler has equal probability of winning and losing the stake.

$$\therefore\ E\left[Y_n\right]=n\left(p-q\right)$$

where $p$ is the probability of winning= $\frac{1}{2}$ and $q$ the probability of losing= $\frac{1}{2}$ . Hence

$$E\left[Y_n\right]=n\left(\frac{1}{2}-\frac{1}{2}\right)=0 \qquad (4.7)$$

It should be noted that

$X_{n+1}=X_0+\sum_{k=1}^{n+1}Y_k=X_0+\sum_{k=1}^{n}Y_k+Y_{n+1}=X_n+Y_{n+1}$, [by equation (4.8)]

$\therefore\ E\left[X_{n+1}\middle|Y_n\right]=E\left[\left(X_n+Y_{n+1}\right)\middle|Y_n\right]$

$=E\left(X_n\middle|Y_n\right)+E\left(Y_{n+1}\middle|Y_n\right)$, [by Theorem 2.2]

$=X_n+E\left(Y_{n+1}\middle|Y_n\right)$, (since $X_n$ is a function of $Y_n$)

$=X_n+E\left(Y_{n+1}\right)$, (since all $Y_n$ are independent)

$=X_n$, since $E\left(Y_{n+1}\right)=0$, [by equation (4.7)].

$\therefore\ \left\{X_n\right\}$ is a martingale.

$\therefore\ E\left[X_n\right]=E\left[X_0\right]$, (by proposition 2.1).

Since $X_0$ is the initial stake, this is a constant,

$$\therefore\ E\left[X_n\right]=X_0 \qquad (4.8)$$

Let $r$ be the ruin probability that $X_n$ reaches $-a$ before reaching $b$ . Hence

$P\left\{X_n=-a\right\}=r$ and

$P\left\{X_n=b\right\}=1-r$ .

It should be noted that $t$ is a stopping time, therefore, the event $\left\{t=n\right\}$ is completely determined by the sequence $\left\{X_n, n\geq 0\right\}$ .

$$\therefore \left.\begin{array}{r} P\left\{X_t=-a\right\}=r \\ and \quad P\left\{X_t=b\right\}=1-r \end{array}\right\} \qquad (4.9)$$

From above,

$E\left(X_t\right)=-aP\left\{X_t=-a\right\}+bP\left\{X_t=b\right\}$ .

$$\Rightarrow E\left(X_t\right)=-ar+b(1-r) . \qquad (4.10)$$

By the **OST**, (4.8) becomes

$E(X_t) = E(X_0) = X_0$.

Using equation (4.10), we have

$E(X_t) = -ar + b(1-r) = X_0$

$\Rightarrow -(a+b)r = X_0 - b$

$$\Rightarrow r = \frac{b - X_0}{(a+b)} \quad (4.11)$$

$\therefore$ This is the ruin probability of the gambler. This proves the first part of the result.

(ii) Let

$$M_n := X_n^2 - n \quad (4.12)$$

$M_n$ is a martingale (by Lemma 2.1)

$\Rightarrow E(M_n) = E(M_0) = E(X_0 - 0) = E(X_0^2)$, (by equation (4.12)

$\Rightarrow E(M_n) = E(M_0)$, [by equation (2.4)]

Since $X_0$ is the initial capital and is constant

$$E(M_n) = E(M_0) = E(X_0^2) = X_0^2 \quad (4.13)$$

Also, using (4.12) in this manner, we have

$E(M_t) = E(X_t^2 - t)$

$\Rightarrow E(M_t) = E(X_t^2) - E(t)$, (by linearity property of expectation) (4.14)

Combining equations (4.13) and (4.14), we have

$$E(M_t) = E(M_0) = E(X_t^2) - E(t) = X_0^2 \quad (4.15)$$

From (4.9), we have

$E(X_t^2) = a^2 P(X_t = -a) + b^2 P(X_t = b) \Rightarrow E(X_t^2) = a^2 r + b^2 (1-r)$

Substituting this in (4.15), we have

$a^2 + b^2 (1-r) - E(t) = X_0^2 \Rightarrow E(t) = ra^2 + b^2 - rb^2 - X_0^2$

$$\therefore E(t) = ra^2 + b^2 - rb^2 - X_0^2 = r\left(a^2 - b^2\right) + b^2 - X_0^2 = \left(b^2 - X_0^2\right) - r\left(b^2 - a^2\right)$$

Using $r = \frac{(b - X_0)}{(a+b)}$ of (4.11) in the above equation, we have

$$E(t) = \left(b^2 - X_0^2\right) - \frac{(b - X_0)\left(b^2 - a^2\right)}{(a+b)} \tag{4.16}$$

$$\therefore E(t) = \frac{(b + X_0)(b - X_0)(a+b) - (b - X_0)(b-a)(a+b)}{(a+b)}$$

$$\therefore E(t) = \frac{(b - X_0)(a+b)\left[b + X_0 - b + a\right]}{(a+b)} \Rightarrow E(t) = (X_0 + a)(b - X_0)$$

Equation (4.16) gives the expected duration of the game. This proves the second part of the result.

**Remark 4.2.1**

- It should be noted that the relation (4.12) ought to be $M_n = X_n^2 - \sigma^2 n$. Here, we have assumed a relation with a unit variance (i.e. $\sigma^2$ =1
- Also, the ruin probability that $X_n$ reaches $-b$ before reaching $a$ is analogous to the above example. The ruin probability of the gambler in this case will be $\frac{(a - X_0)}{(a+b)}$ and the expected duration of the game will be $(a - X_0)(X_0 + b)$.
- It can be observed that the random walk starts from $X_0$ in the above example. If however the walk starts from 0 instead of $X_0$, then the ruin probability that $X_n$ reaches $-a$ before reaching $b$ is $\frac{b}{a+b}$ and the expected duration of the game becomes $ab$.

- In the above procedure for the proof, we considered the accumulated fortune of the gambler which satisfies martingale property. This gives rise to the result of proposition 2.1. The optional stopping theorem was then applied to determine the ruin probability and the expected duration of the game. Hence we conclude that optional stopping theorem is an indispensable tool in this determination. It should also be noted that the ruin probability that $X_n$ reaches $-b$ before $a$ can also be determined in the same manner. This becomes $\frac{a}{(a+b)}$ and the expected duration of the game is also $ab$.

## 5.0 Conclusion

The result obtained from this study shows that the determination of ruin probability and the expected duration of a coin flipping game using both Wald’s first two identities (a direct approach) and Optional Stopping Theorem gives rise to the same result. However it should be noted that the Optional Stopping Theorem tells us that even with a well –chosen strategy for stopping a game, under some reasonable hypotheses, a martingale is still a fair game.

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