UNCERTAINTY CLAIM PRICING USING WEIBULL DISTORTION OPERATOR

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ABSTRACT

In this paper, we consider the problem of uncertainty Claim Pricing using Distortion Operators. This approach was first developed in Insurance Pricing, where the original distortion function was defined in terms of the normal distribution. We generalize this approach by using a distortion that is based on the Weibull distribution. The Weibull family allows for heavier and skewed tail because it is so flexible that other statistical distributions can be recovered from it by change of parameters. The problem of uncertainty Claims has been extensively studied for non-Gaussian model in which the formula was derived for the NIG asset pricing. We show here how Weibull based distortion function can be used to derive the formula for asset pricing of uncertainty future returns of a risky asset. We derive also the risk measure for the incurred risk modeled by the Weibull variables and show that it follows the power law.

Key words: Uncertainty claim, Weibull distribution, Insurance pricing, Power law.

MSC: 91G10

INTRODUCTION

Distortion Risk Measures is used to price financial and insurance risk and estimate the returns of most financial assets. This is done by adjusting the true probability to give more weight to higher risks events. They are used to determine the future price denoted by ST, in respect of a random loss variable$x$, with the view to avoiding insolvency. The future returns for the risky asset is the expected value under the distorted probability which is called the risk adjusted measure. Distortion risk measures predict the prize of return for a given risk portfolio, based on its downside risk potentials.

A distortion risk measure can be defined as the distorted expectation of any non-negative loss random variable X. It is accomplished by using a “dual utility” or the distortion function g as follows;

$$P(g(x)) = \int_0^\infty g\,(1 - f_x(x))\; dx = \int_0^\infty g\, S_x(x) dx\,, \qquad (1)$$

where;g: $[0,1] \rightarrow [0,1]$ is a continuous increasing function with,

$$g(0) = 0 \; and \; g(1) = 1.$$

$F_x(x)$denotes the cumulative distribution function of X, and$g(F_x(x))$ is the distorted distribution function.

The survival function is given as;

$S_x(x) = 1 - f_x(x) = \mathrm{P}\,(x > x)$ (2)

Properties of Distortion Risk Measures [1]

The properties of a distortion risk measure are given below;

1) Monotocity; if x ≥0,then$P(g(x)) \geq 0$.
2) Positive homogeneity;$(g(\lambda x)) = \lambda P(g(x))$,$\forall \lambda \geq 0$.
3) Translate Invariance; $(g(x + c)) = P(g(x)) + c$,$\forall\; c \in R$.
4) $P(g(-x)) = -P(\bar{g}(x))\; where\; \bar{g}(x) = 1 - g(1 - x)$
5) If a random variable $x_n$has a finite number of values that is $(x_n \overset{w}{\rightarrow} x)$ and if $P(g(x))$ exists, then$P(g(x_n)) \rightarrow P(g(x))$.
6) ComonoticAdditivity; if x and y are comonotonic risks taking positive and negative values then,
   $P(g(x + y)) = P(g(x)) + P(g(y))$.
7) In the general cases, distortion risk measures are not additive
   $$P(g(x + y)) \neq P(g(x)) + P(g(y)).$$
8) Sub-additivity; if the distortion function $g(x)$ is concave,then
   $P(g(x + y)) \leq P(g(x)) + P(g(y))$.
9) For a non-decreasing distortion function g, the associated risk measure$P(g)$ is consistent with the stochastic dominance of order one. That is if$x \leq_1 y$,then $P(g(x)) \leq P(g(y))$.
10) For a non-decreasing concave distortion function g, the associated risk measure $P(g)$ is consistent with the stochastic dominance of order two. That is, if $x \leq_2 y$then
    $$P(g(x)) \leq P(g(x)).$$
11) For a strictly concave distortion function g, the associated risk measure $P(g)$is strictly consistent with the stochastic dominance of order 2,if for
    $x \leq_2 y$then $\mathrm{P}g(x) < \mathrm{P}g(y)$.

Example of Distortion Risk Measure

The mathematical expectation of $g(x) = x$if it exists is given as;

$$E(x) \;\; = Pg(x) \;\; , \qquad\qquad (3)$$

and the Varince at risk (VaR) corresponds to the distortion given as follows;

$$g(x) = \begin{cases} 1 \;\; , if u < \alpha \\ 0 \; , \; if \; u \geq \alpha \end{cases} (4).$$

Figure 1 show this relationship.

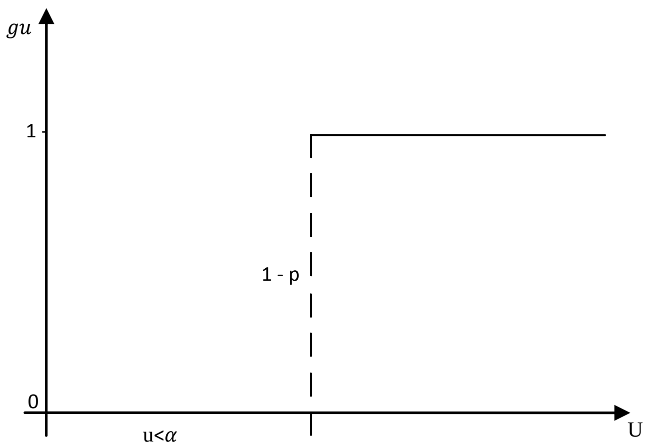

Figure 1: Distortion function as a step function.

As shown in fig.1, the distortion function is a step function with discontinuity at $1 - \alpha$ and a big jump at $u = \alpha$. This discontinuity shows that VaR is not a coherent risk measure. The value $1 - \alpha$ is the ruin or the expected shortfall.

The conditional value at risk (CVaR) corresponds to the distortion;

$$g(x) = \begin{cases} 1 \;, \quad if \quad u < \alpha \\ \frac{u-\alpha}{1-\alpha} \;, \; if \quad u \geq \alpha \end{cases} \qquad . \qquad\qquad (5)$$

This is shown in the figure 2.

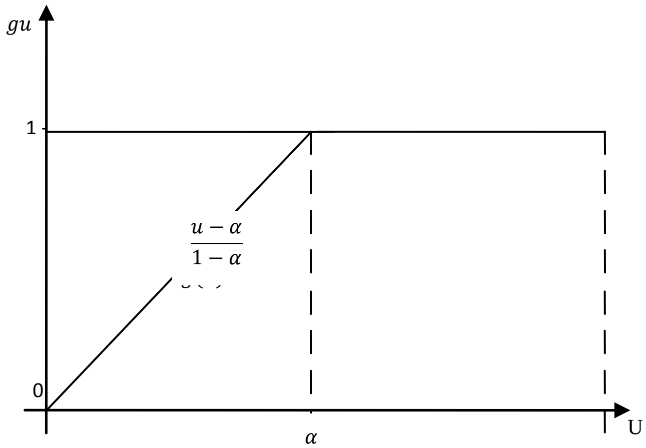

Figure2: The conditional Value at risk as distortion function.

The conditional value at risk (CVaR) as depicted is continuous but not differentiable at u = α because of its zero value at this point. The distortion function g maps all percentiles below $\alpha$to a simple point 0. This means that all information contained in that part of the distribution is lost.Any smooth differentiable distortion g will give a coherent risk measure that is different from CVaR.

A class of distortion based on the normal distribution in order to know the price for future returns of most risk assets was proposed as follows;

$$Pg(x) = \int g\, S_x(x)dx = \emptyset(\emptyset^{-1}(u) + \alpha) \qquad (6)$$

where,$\emptyset$ is the standard normal cumulative distribution function[2].The pricing kernel associated to the distortion is;

$$H[x = h(z){:}\,\alpha] = \int_{\infty}^{0} g_\alpha S_x dx \ , \qquad (7)$$

and$E[h(z + \alpha)] = S_0 \mathrm{e}^{z+\alpha}$. (8)

h is a continuous positive and increasing function. Clearly, for a normal random variable z,

$H[x = h(z){:}\,\alpha] = E[h(z + \alpha)]$(9)

and

$$H[x = h(z){:} -\alpha] => H[ST - \alpha] = S_0 \mathrm{e}^{z_T + \alpha_T} \qquad (10)$$

Black-Schole modeled the risk position $S_t$of an asset at timeby a geometric Brownian motion as;

$S_t = S_0 e^{\left(\mu-\frac{\theta^2}{2}\right)t}$ to w_t , $t \geq 0$ (11)

where w is the Brownian motion.

For the pay-off of a European call option (with maturity T and strike price k) we have

$$S_T = C(S_T, k) = (S_T - k), \tag{12}$$

where S_T is a lognormal random variable.

Applying the kernel to this payoff with;

$$\alpha = \frac{u-rc\sqrt{T}}{\sigma}, \tag{13}$$

itcan be shown that;

$$e^{-rT}H[C(X_T,k) := S_0\emptyset(\ln\left(\frac{x_0}{k}\right)) + \left(\frac{r+\theta^2}{2}\right)T - e^{-rT}k\phi\frac{\ln\left(\frac{x_0}{k}\right)+\left(\frac{r+\theta^2}{2}\right)T}{\sigma\sqrt{}} - \sigma\sqrt{T}$$

$$= S_0\phi[\ln\left(\frac{x_0}{k}\right)+\left(\frac{r+\theta^2}{2}\right)]T - e^{-rT}k\phi\frac{\ln\left(\frac{x_0}{k}\right)+\left(\frac{r+\theta^2}{2}\right)T-\sigma\sqrt{T}}{\sigma\sqrt{T}} \quad . \tag{14}$$

The risk free rate$r$, shows that the Black-Scholes formula can be recovered from the distortion operator[2].

The Weibull distribution

We now introduce a generalized version of the distortion that is based on a Weibull distribution.

Definition The Weibull distribution function is given by;

$$f(x) = f(x, k, \beta, \alpha) = \frac{k}{\beta}\frac{(x-\alpha)^{k-1}}{\beta}e^{\frac{-(x-\alpha)^k}{\beta}} \tag{15}$$

This consists of the failure and reliability rates[3].x is the loss random variable and α, β , are the scale parameters and k the shape parameter. The shape parameters determine the properties of the rate of return and the scale parameter $\beta > 0$ is proportional to the mean time failure.

Firstly, we compute the empirical survival function S_x, then, the Weibull distortion operator is used to compute;

$P(g(x)) = \int_0^\infty g\, S_x(x)dx.$

We define the Web distribution operator as

$$g_{x,k,\beta,\alpha}(u) = \emptyset^w[\emptyset^{w-1}(u) + \lambda] \qquad (16)$$

The same calibrationas in Wang distortion is adopted; firstly, if a random variable has a Weibull distribution it's probability density function (pdf) is ;

$$f(x) = f(x,k,\beta,\alpha) = \begin{cases} \frac{k}{\beta}\left(\frac{x-\alpha}{\beta}\right)^{k-1} e^{\frac{-(x-\alpha)^k}{\beta}}, x \geq 0 \\ 0 \qquad\qquad x < 0 \end{cases} \qquad (17)$$

where, $k > 0, \alpha > 0$.When $\alpha = 0$ it reduces to two- parameter Weibull distribution.The future return of the risky asset at S_T,is assumed to follow a Weibull distribution instead of the normal distribution.

$$S_T = g(z_T), \qquad (18)$$

is the price of the security at time T.

For a function $g_u = S_0 e^u$ of a random variable Z_T with distribution$Web(x,k,\beta,\alpha)$implies $f(x,k,\beta,\alpha)$.

$$H(S_T,\lambda) = \emptyset^w[\emptyset^{w-1}(u) + \lambda] = g_{x,k,\beta,\alpha} S_x(x)dx, \qquad (19)$$

where$\emptyset^w$ is the Weibull distribution.

Proposition 1

Let Z be a random variable and let $x = g(z)$ be the transformation of the continuous function$f(x)$ and $\emptyset^w = f(x,k,\beta,\alpha) =$ Web (x,k,β,α) implies $f(x)$.

Given that

$S(x) = 1 - f(x) = P(x > t)$or

$$S(x) = 1 - \text{Web}\ (x,k,\beta,\alpha)$$

$= 1 - \emptyset^w(x,k,\beta,\alpha).$

Then,

$$H(X,\alpha) = \int_0^\infty g_\alpha\, S_x(x)dx = E[g(z + \lambda\beta)] \qquad (20)$$

Proof

Define;$H(x,\alpha) = \int_0^\infty g_\alpha \, S_x(x)dx = \emptyset^w[\emptyset^{w-1}(u) + \lambda]$,where

$$S(x) = P(x > t)$$
$$= P(g(z) > t)$$
$$= P(\frac{g(z)}{g} > {}^{t}/_{g})$$
$$= P\big(z > g^{-1}(t)\big)$$

$= 1 - \emptyset^w(g^{-1}(t))$
$= \emptyset^w[-g^{-1}(t)]$
= U. (21)

Applying (21) to the distortion function, we obtain;

$$g_{(x,k,\beta,\alpha)}S_x(t) = \emptyset^w[\emptyset^{w-1}(u) + \lambda] \;\; \emptyset^w[\emptyset^{w-1}(\emptyset^w[-g^{-1}(t)]) + \lambda]$$
$$= \emptyset^w[-g^{-1}(t) + \lambda]$$

$= 1 - \emptyset^w(g^{-1}(t) - \lambda)$ (22)

The second application of probability and normalizing gives;

$$P[z > g^{-1}(t) - \lambda] \;=\; P\left[\frac{z-\alpha}{\beta} > \frac{g^{-1}(t) - \lambda - \alpha}{\beta}\right]$$
$$= p\left[\frac{z-\alpha}{\beta} > \frac{g^{-1}(t) - \lambda - \alpha}{\beta}\right]$$
$$= P(x > \frac{g^{-1}(t) - \lambda - \alpha}{\beta})$$

$= P(x\beta > g^{-1}(t) - \lambda - \alpha\beta)$ (23)

Note: if $x = \frac{z-\alpha}{\beta}$

$$x\beta = z - \alpha$$

$z = x\beta + \alpha.$

Applying the above,we get;

$$P(z\beta + \alpha + \alpha\beta) > g^{-1}(t))$$

$P[z + \alpha\beta > g^{-1}(t)]$. (24)

Multiplying through by g, we have;

$$P[g(z+\alpha\beta) > g^{-1}(t).g] = P[g(z+\alpha\beta > t]$$

$= E^n[g(z+\alpha\beta)$. (25)

Therefore

$E^n{}_x = E^n\big(g(z+\alpha\beta)\big) = \int g_\alpha S_x(x)dx = E[S_0 e^{2+\alpha\beta}]$, (26)

whereE^n denotes the expectation under the density measure n.So

$H(S_T - \lambda) = E\big[S_0 e^{2+\alpha\beta} T\big]$ (27)

This result shows that under a $g_{x,k,\beta,\alpha}$ distortion, a Weibull random variable is translated by a factor $\lambda\beta$. This generalizes the equivalent result found in NIG (Normal Inverse Gaussian Distribution).

This simplifies $G[S_T\ - \lambda]$ to;

$G(S_T - \lambda) = S_0 e^{rT}$ (28)

Under Weibull distortion with value λ,$\emptyset^w[\emptyset^{w-1}(u) + \lambda]$, the price at ST evolves like a risk neutral asset because λ is calibrated to verify the risk neutral condition.

Hence,$H(S_T - \lambda) = S_0 e^{rT}$

$$= \int g_{xk\beta\alpha} S_x(t)dt$$

$$= \emptyset^w[\emptyset^{w-1}(u) + \lambda]$$

$$= E\big[S_0 e^{2+\lambda\beta}\big]$$

$= E[g(z + \lambda\beta)]$,

as required. This implies;

$G[S_T - \lambda] = E\big[S_0 e^{2t-\lambda\beta} T\big]$. (29)

The positivity of g in $E[z + \lambda\beta]$ can be relaxed. More generally, the proposition can be extended to the case when the security price is a function of a symmetrically distributed random variable. Apply the price $E[z + \lambda\beta]$, the capital required for the standard European call payoff is

$G[F(ST,k);\lambda] = E[g(2T + \lambda\beta]$ (30)

$= \left[S_0 e^{2t+\lambda\beta} - k\right]$

$$= \int_{-\infty}^{\infty} (S_0 e^2 - k)\, web(x{:}\, k, \beta, \alpha + \lambda\beta) dt. \quad (31)$$

Having known the density function $G(F(S_T))$ of the stock price at the expiring time ST under the risk neutral measure $\lambda\beta$, we can easily prize European call and put option by simply calculating the expected value[4].

Hence, for a European call option with strike price k at time S_t to expiration S_T, the value at time 0 is therefore given by the expectation of the payoff under the Martingale measure.

$$G(k, T) = E_Q[exp^{(-rT)} \max(ST - k), 0]$$

$$= exp^{(-rT)} \int_0^{\infty} g_Q(S, T) \max(S, -K) ds$$

$$= e^{-rT} \int_k^0 g_Q(S, T)(S - k) ds$$

$$= exp^{(-rT)} \int_0^{\infty} g_Q(ST) s\, ds - k(exp^{-rT})\, \pi. \quad (32)$$

whereπ is the probability under Q of the future prize of the risk asset,

$$G(k, T) = \int_{Q\alpha}^{\infty} (S_0 e^z - k) \frac{k}{\beta} z^{k-1} e^{-zk} + \alpha\beta) dz. \quad (33)$$

then,

$$G[F(S_T, k) - \lambda] = \int_{Q\alpha}^{\infty} [(S_0 e^z - k) k z^{k-1} e^{-zk} - \lambda)] dz$$

$$= \int_{Q\alpha}^{\infty} [S_0 e^z . k z^{k-1} e^{-zk} - \lambda] dz - k \int_{Q\alpha}^{\infty} [k z^{k-1} e^{-z} - \lambda] dz \quad (34)$$

If $\lambda = \frac{\mu - rc}{\beta} T$, (35)

$S_T$evolve as a risk neutral asset.

This shows that the price evaluated with the pricing kernel associated to the Weibull distortion with parameter λ is given by;

$$G[F(ST, k) - \lambda] = S_0 e^z \int_{Q\alpha}^{\infty} web\, [x, k, \beta_T(\alpha + \lambda) T\, dz - k \int web(x, k, \beta_T(\alpha + \lambda)\, T dz$$

$$= S_0 e^z (1 - Q_\alpha^w - \lambda) - (1 - Q_\alpha^w - \lambda)$$

$$= S_0 e^z(Q_\alpha^w + \lambda) - k(Q_\alpha^w + \lambda)$$

(36)$e^{-rT}G(F(ST\ k), -\lambda) = S_0\ web[Q_\alpha, x, k, \beta(\alpha + \lambda)]$

$-ke^{-rT}\ web[Q_\alpha: x, k, \beta_T(\alpha + \lambda)T$(37)

which is the prize at the time t = 0 .That is the price at S_t.

Theorem 1

If Z_t has the Weibull distribution and a power function, $g(Z) = Z^\beta$, then, the optimal strategy has the power law distribution given by

$H_\beta(Z) = \frac{Z^{1-\beta}}{\gamma\beta}$ (39)

Proof

Here, $Z_t$is the continuous returns of prices of a security. Mataz[5] had shown that it is possible to find an optimal investment strategy in terms of the probability density function describing the prices returns of a security. This strategy optimizes some appropriate measure of risk. Hence given our assumption of Weibull distribution of asset returns, we define the strategy that optimizes the variance of the return distribution as;

$H_\beta(Z) = \int_0^\infty g(Z)dZ,$ (40)

where$g(Z)$ is as in (15). Equation (40) reduces to (using the transformation in[6] and [7])

$H_\beta(Z) = \int_0^\infty exp\left[-\gamma Z^\beta\right] dZ.$ (41)

Set $x = \gamma Z^\beta$, then $\frac{dx}{dZ} = \gamma\beta Z^{\beta-1}$ or $dZ = \frac{1}{\gamma\beta} Z^{1-\beta} dx$. Substitute into (41) to get

$$H_\beta(Z) = \frac{1}{\gamma\beta} Z^{1-\beta} \int_0^\infty exp[-x]\, dx$$

$= \frac{1}{\gamma\beta} Z^{1-\beta},$ (as required).

Conclusion

Under Weibull distortion with value $\lambda = \frac{\mu - rc}{\beta}$T,ln $\emptyset^w[\ \emptyset^{w-1}(u) + \lambda]$ the prize at future timeS_T evolves like a risk neutral asset. λis calibrated to verify the risk neutral condition.

The assumption that the prize of return at S_T follows a Weibull distribution in place of normal distribution is based on its flexibility to follow the behavior of other statistical distribution such as the exponential and normal distribution. That is, other statistical distributions can be recovered from the Weibull distribution by change of the values of the parameters. From (39), $(1-\beta)$ is the characteristic exponent of the generalized power law distribution. This shows that power law property relation characterizes the measure of risks modeled by Weibull variables. (39) is the incurred risk measure of an investor faced with investment decision.

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