A Generalization of Sufficient Conditions for Quasiconcavity of Twice Differentiable Real-Valued Single-Variable Functions

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Abstract

This paper considers the characterizing properties used in the definition and recognition of real-valued single-variable quasiconcave functions. Through a presentation of the link between the first and second order conditions for quasiconcavity, it presents a characterization which unifies these various characterizing properties. Thus it provides an equivalence for the various definitions of quasiconcavity.

Keywords: *Quasiconcavity, Level Sets, Line Segment Minimum Property, Twice Differentiable Function.*

1 Introduction

As generalizing as analyses on $\mathbb{R}^n$ may look, it is necessary to consider works on $\mathbb{R}$ especially if they form the bases for works on $\mathbb{R}^n$. Further the complexity and intimidating nature of works presented in $\mathbb{R}^n$ leave the unexposed and unexperienced handicapped and confused. This work attempts to solve such problems for quasiconcave functions. It presents a characterization of real-valued single-variable quasiconcave functions. This characterization unifies the various characterizing properties of quasiconcave functions.

A quasiconcave function is a real-valued function defined on an interval or a convex subset of a real vector space such that the inverse image of any set of the form $(-\infty, a)$ is a convex set. All convex functions are quasiconcave, but not all quasiconcave functions are concave. So quasiconcavity is a generalization of concavity.

Quasiconcave functions need not necessarily be differentiable; however, this work considers differentiable real-valued single-variable quasiconcave functions.

Definition 1.1 *If* $x, y \in \mathbb{R}$ *and* $\lambda \in [0,1]$, *then* $\lambda x + (1+\lambda)y$ *is a convex combination of* x *and* y. Geometrically, a convex combination of x and y is a point somewhere between x and y.

Definition 1.2 *A set* $I \subseteq \mathbb{R}$ *is convex if* $x, y \in I$ *implies* $\lambda x + (1+\lambda)y \in I$ *for all* $\lambda \in [0,1]$. The definition of a convex set immediately implies that I is convex if and only if I is either empty, a point, or an interval.

Throughout this work we suppose that I is a convex subset of $\mathbb{R}$.

Definition 1.3 $f: I \longrightarrow \mathbb{R}$ *is concave if for any* $x, y \in I$, *we have, for all* $\lambda \in (0,1)$,

$$f(\lambda y + (1+\lambda)x) \geq \lambda f(y) + (1+\lambda)f(x) \tag{1.1}$$

$f: I \longrightarrow \mathbb{R}$ *is strictly concave if for any* $x, y \in I$, *with* $x \neq y$, *we have, for all* $\lambda \in (0,1)$,

$$f(\lambda y + (1+\lambda)x) > \lambda f(y) + (1+\lambda)f(x) \tag{1.2}$$

In words, a function is concave if its value at the linear combination between two points in its domain is greater than or equal to the weighted average of the function's values at each of the points considered
In practical terms, the difference is that concavity allows for linear segments, but strict concavity does not. Concavity allows for ascending and descending linear segments. Vertical segments are excluded because of $x \neq y$. Horizontal segments are excluded, because such lines would allow chords to be drawn above the curve, violating the requirements of equation (1.1).

2 Quasiconcave functions

Definition 2.1 *A function* $f: I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ *is called quasiconcave if its domain and all its superlevel sets*

$$I_\beta = \{x \in dom\, f \mid f(x) \geq \beta\}, \tag{2.1}$$

for $\beta \in \mathbb{R}$ *are convex.*

A function is *quasiconvex* if $-f$ is *quasiconcave*, that is, every *sublevel set* $\{x \in dom\, f \mid f(x) \leq \beta\}$ is convex. A function that is both quasiconcave and quasiconvex is called quasilinear. A function is quasilinear if its domain, and every level set $\{x \mid f(x) = \beta\}$ is convex.

Quasiconcavity requires that each sublevel set be an interval (including, possibly, an infinite interval).

2.2 Examples of Quasiconcave Functions

The following are examples of quasiconcave functions on $\mathbb{R}$:

(i) Define $f: \mathbb{R}_+ \longrightarrow \mathbb{R}$ by

$$f(x) = \begin{cases} 0 & \text{if } \; 0 \leq x \leq 1, \\ (x-1)^2, & \text{if } \; x > 1. \end{cases} \tag{2.2}$$

Since f is non-decreasing, it is quasiconcave.

(ii) Define $f: \mathbb{R}_+ \longrightarrow \mathbb{R}$ by

$$f(x) = \begin{cases} x^3, & \text{if } \; 0 \leq x \leq 1, \\ 1, & \text{if } \; 1 < x \leq 2, \\ x^3 & \text{if } \; x > 2\,. \end{cases} \tag{2.3}$$

Since f is non-decreasing, it is both quasiconcave and quasiconvex on $\mathbb{R}$. But f is discontinuous at $x = 2$. Moreover, f is constant on $(1, 2)$, and hence every point in this open interval is a local maximum as well as a local maximum. However, no point in $(1, 2)$ is either a global maximum nor a global minimum. Finally, $f(0) = 0$, but 0 is neither a local maximum nor a local minimum.

(iii) The logarithm function

$$f(x) = log\, x \tag{2.4}$$

on $\mathbb{R}_{++}$ is quasiconcave (and quasiconvex, hence quasilinear).

(iv) The Ceiling function

$$ceil(x) = inf\, \{z \in Z | z \leq x\} \tag{2.5}$$

is quasiconcave (and quasiconvex)

These examples show that quasiconcave functions can be discontinuous [1].

We can give a simple characterization of quasiconcave functions on $\mathbb{R}$. We consider continuous functions, since stating the conditions in the general case is cumbersome. *A continuous function* $f: \mathbb{R} \longrightarrow \mathbb{R}$ *is quasiconcave if, and only if, at least one of the following conditions holds:*

- f *is nonincreasing*
- f *is nondecreasing*

- *there is a point $c \in dom\ f$ such that for $t \leq c$ (and $t \in dom\ f$), f is nondecreasing, and $t \geq c$ (and $t \in dom\ f$), and f is nonincreasing* [1]

3. The Line Segment and Local Minimum Property

It is quite difficult to get simple necessary and sufficient conditions for quasiconcavity in a case where f is twice continuously differentiable. Thus we will need the following definitions

Definition 3.1 *Let $I \subseteq \mathbb{R}$ be a nonempty open interval, then $f: I \to \mathbb{R}$ has the line segment minimum property if and only if for $x, y \in I$, $x \neq y$,*

$$min_{\alpha} \{f(\alpha x + (1-\alpha)y):\ 0 \leq \alpha \leq 1\} \tag{3.1}$$

exists.

That is, the minimum of f along any line segment in its domain of definition exists.

It is easy to verify that if f is a quasiconcave function defined over the interval I, then it satisfies the line segment minimum property (3.1), since the minimum will be attained at one or both of the endpoints of the interval; that is, the minimum will be attained at either $f(x)$ or $f(y)$ (or both points) since $f(\alpha x + (1-\alpha)y)$ for $0 \leq \alpha \leq 1$ is equal to or greater than $\min\{f(x), f(y)\}$ and this minimum is attained at either $f(x)$ or $f(y)$ (or both points).

Definition 3.2 *Let $\alpha \in \mathbb{R}$, the function h defined over an interval I' attains a semistrict minimum at $t_0 \in Int\ I'$ if and only if there exist $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ such that $\alpha_0 - \varepsilon_1$, $\alpha_0 + \varepsilon_2 \in I$ and*

$$h(\alpha_0) \leq h(\alpha) \tag{3.2}$$

for all α such that $\alpha \in [\alpha_0 - \varepsilon_1,\ \alpha_0 + \varepsilon_2]$;

$$h(\alpha_0) < h(\alpha_0 - \varepsilon_1) \tag{3.3}$$

and

$$h(\alpha_0) < h(\alpha_0 + \varepsilon_2) \quad [2]. \tag{3.4}$$

If h just satisfies (3.2) at the point α_0, then it can be seen that it attains a *local minimum* at α_0. But conditions (3.3) and (3.4) show that a *semistrict local minimum* is stronger than local minimum: for h to attain a semistrict local minimum at such α_0, we need h to attain a local minimum at such α_0, but the function must eventually strictly increase at the end points of the region where the function attains the local minimum. Note that h attains a *strict local minimum* at such $\alpha_0 \in Int\ I'$ if and only if there exists $\varepsilon > 0$ such that $\alpha_0 - \varepsilon,\ \alpha_0 + \varepsilon \in I'$ and

$$h(\alpha_0) < h(\alpha) \tag{3.5}$$

for all α such that $\alpha_0 - \varepsilon \leq \alpha \leq \alpha_0 + \varepsilon$ but $\alpha \neq \alpha_0$.

It can be seen that if h attains a strict local minimum at α_0, then it also attains a semistrict local minimum at α_0. Hence, a semistrict local minimum is a concept that is intermediate to the concept of a local and strict local minimum [3, 4, 5].

4. Characterizations of Quasiconcave Functions

The following characterizations will be helpful in the main result in section 6.

Theorem 4.1 The Minimum Function Value Test Characterization of Quasiconcave Functions: *Let $I \subseteq \mathbb{R}$ be a closed interval, a function $f: I \to \mathbb{R}$ is quasiconcave if and only if for $x, y \in I$ and $\lambda \in [0,1]$*

$$f(\lambda x + (1-\lambda)y) \geq \min\ \{f(x),\ f(y)\}. \tag{4.1}$$

The above result means that the line segment joining x to y that has height equal to the minimum value of the function at the point x and y lie below (or is coincident with) the graph of f along the line segment joining x to y [6, 7, 8, 9, 10]. This is a variant of Jensen's inequality that characterizes quasiconcavity.

If f is concave over I, then

$$f(\lambda x + (1-\lambda)y) \geq \lambda f(x) + (1-\lambda)f(y) \qquad \text{(by (1.2))} \tag{4.2}$$

$$\geq \min\ \{f(x), f(y)\}, \tag{4.3}$$

where the second inequality results from the fact that $\lambda f(x) + (1-\lambda)f(y)$ is an average of $f(x)$ and $f(y)$. Thus if f is concave it is also quasiconcave. This characterization can be written in an equivalent form as shown in the next result, thus we can use them interchangeably.

Theorem 4.2 The Function Value Comparison Characterization of Quasiconcavity

Let $\emptyset \neq I \subseteq \mathbb{R}$ be open, then $f: I \to \mathbb{R}$ is quasiconcave if and only if for $x, y \in I$, $x \neq y$

$$f(y) \geq f(x) \Longrightarrow f(\lambda x + (1-\lambda)y) \geq f(x) \ \text{[11, 12].} \tag{4.4}$$

This means that the function f is quasiconcave if $f(y) \geq f(x)$ implies that its value at a convex combination of two points in its domain is greater than or equal to $f(x)$ which is the function value of the smaller function value of the two points.

4.3 The Derivative-Based Characterization of Quasiconcavity

Theorem 4.3a **First Order Condition**: *Let $f: I \to \mathbb{R}$ be a once differentiable function defined over the open interval $I \subseteq \mathbb{R}$, $x, y \in I$, $x \neq y$; then f is quasiconcave if and only if*

$$f'(x)(y-x) < 0 \Rightarrow f(y) < f(x). \tag{4.5}$$

In this characterization we assumed that I is open and that the derivatives of f exist and are continuous functions over I [11, 13].

Now consider the following result which is contrapositive to Theorem 4.3a making them (Theorem 4.3a and Corollary 4.3b) logically equivalent.

Corollary 4.3b **First Order Condition**: *Let $I \subseteq \mathbb{R}$ be an open interval in $\mathbb{R}$ and suppose $f: I \to \mathbb{R}$ is a once differentiable function, then f is quasiconcave if and only if*

$$f(y) \geq f(x) \Rightarrow f'(x)(y-x) \geq 0, \quad x, y \in I, \quad x \neq y \text{ [7]}. \tag{4.6}$$

4.4 Line Segment Minimum Property Characterization of Quasiconcavity

Theorem 4.4 *Suppose $f: I \subseteq \mathbb{R} \to \mathbb{R}$ has the line segment minimum property for $x \in I$, then f is quasiconcave if and only if*

$$x, y \in I, x \neq y \implies h(\alpha) \equiv f(x + \alpha(y-x)) \tag{4.7}$$

does not attain a semistrict local minimum for any α such that $0 < \alpha < 1$.

4.5 The Derivative-Based Characterization of Quasiconcavity-Second Order Condition

Theorem 4.5: *Let $I \subseteq \mathbb{R}$ be a nonempty open interval. Then $f: I \to \mathbb{R}$ a twice differentiable function is quasiconcave if and only if for $x, y, z \in I$, with $y \neq x$*

$$\begin{gathered} f'(z)(y-x) = 0 \quad \Rightarrow (i)\ f''(z)(y-x)^2 < 0 \ \text{ or } \ (ii)\ f''(z)(y-x)^2 = 0 \\ \text{and} \\ h(\alpha) = f(z + \alpha(y-x)) \end{gathered} \tag{4.8}$$

does not attain a semistrict local minimum at $\alpha = 0$ [2].

4.6 Upper Level Set Characterization of Quasiconcave Functions

Theorem 4.6: *Let $I \subseteq \mathbb{R}$ be a nonempty open interval. The function $f: I \to \mathbb{R}$ is quasiconcave if and only if for every $\beta \in Range\ f$ the upper level set*

$$I_\beta = \{x \in dom\ f \mid f(x) \geq \beta\} \tag{4.9}$$

is convex.

5. A Characterization of Quasiconcave Functions through the First and Second Order Conditions

Next we present a result which characterizes quasiconcavity through the first and second order conditions without necessarily involving any of the conventional definitions. It forms a part of the proof for the result in the next section.

Theorem 5.1 *Let $f: I \subseteq \mathbb{R} \to \mathbb{R}$ be a twice differentiable quasiconcave function and let I be nonempty and open, then the following statements are equivalent.*

(*i*) $f'(x)(y-x) < 0 \implies f(y) < f(x), \; x < y.$ (5.1)

(*ii*) $f'(x)(y-x) = 0 \implies f''(x)(y-x)^2 < 0$ (5.2)

and

$$h(\alpha) \coloneqq f(x + \alpha(y-x)) \tag{5.3}$$

does not attain a semistrict local minimum at $\alpha = 0$.

***Proof*:** Suppose

$$f'(x)(y-x) < 0 \implies f(y) < f(x), \quad x < y, \tag{5.4}$$

then

$$\theta f'(x)(y-x) + f(y) < f(x), \quad \theta \in [0,1] \tag{5.5}$$

Since f is twice differentiable at x, we have

$$f\big(x + \theta(y-x)\big) = f(x) + \theta(y-x)f'(x) + \tfrac{1}{2}\theta^2(y-x)^2 f''(x) \tag{5.6}$$

From (5.5) and (5.6) we have

$$2\theta f'(x)(y-x) + f(y) + \tfrac{1}{2}\theta^2(y-x)^2 f''(x) < f\big(x + \theta(y-x)\big) \tag{5.7}$$

Since $f'(x)(y-x) = 0$, (5.7) becomes

$$\implies \tfrac{1}{2}\big(\theta(y-x)\big)^2 f''(x) < f\big(y + \theta(y-x)\big) \tag{5.8}$$

Dividing through by θ^2 and letting $\theta \longrightarrow 0$, we have that

$$(y-x)^2 f''(x) < 0$$

Furthermore, since $f(y) < f(x)$ means that f attains a minimum at y with $x \neq y$ it follows that $x - y \neq 0$ so that $f'(x)(y-x) < 0 \implies f'(x) \neq 0.$

Thus

$$h(\alpha) = f(x + \alpha(y - x))$$

does not attain a semistrict local minimum at $\alpha = 0$.

Conversely, suppose $f'(x)(y-x) = 0 \implies f''(x)(y-x)^2 < 0$ and $h(\alpha) \coloneqq f(x + \alpha(y-x))$ does not attain a semistrict local minimum at $\alpha = 0$, it follows by Taylor's expansion

$$f\big(x + \theta(y-x)\big) = f(x) + \theta(y-x)f'(x) + \tfrac{1}{2}\theta^2(y-x)^2 f''(x) \tag{5.9}$$

that

$$f\big(x + \theta(y-x)\big) - f(x) = \tfrac{1}{2}\theta^2(y-x)^2 f''(x) < 0 \text{ and } f(x) > f(y) \tag{5.10}$$

$$\implies f\big(x + \theta(y-x)\big) - f(x) + f(y) + f(x) < 0 \tag{5.11}$$

Thus

$$f\big(x + \theta(y-x)\big) - f(x) < 0 \implies f(y) < f(x) \tag{5.12}$$

and hence from (5.9)

$$f'(x)(y-x) < 0 \implies f(y) < f(x). \quad \blacksquare \tag{5.13}$$

6. Sufficient Conditions for Quasiconcavity of Twice Differentiable Real-Valued Single Variable Functions

Theorem 6.1 *Let* $f: I \longrightarrow \mathbb{R}$ *be a twice differentiable function over the open interval* $I \subseteq \mathbb{R}$*, with* $x, y \in I,\ x \neq y,\ \lambda \in (0,1)$. *Suppose* f *has the line segment minimum property, then the following statements are equivalent:*

(*i*) $I_\beta = \{x | f(x) \geq \beta,\ \ x \in I,\ \ \beta \in Range\ f\}$ (6.1)

is convex.

(*ii*) $f(y) \geq f(x) \ \Longrightarrow f(\lambda x + (1-\lambda)y) \geq f(x).$ (6.2)

(*iii*) $f'(x)(y-x) < 0 \ \Longrightarrow f(y) < f(x),\ x < y.$ (6.3)

(*iv*) $f'(x)(y-x) = 0 \ \Longrightarrow f''(x)(y-x)^2 < 0$ (6.4)

and

$h(\alpha) \coloneqq f(x + \alpha(y-x))$ (6.5)

does not attain a semistrict local minimum at $\alpha = 0$.

(*v*) $h(\alpha) \coloneqq f(x + \alpha(y-x))$ (6.6)

does not attain a semistrict local minimum for any α *such that* $0 < \alpha < 1$.

(*vi*) $f(\lambda x + (1-\lambda)y) \geq min\ \ \{f(x),\ f(y)\}.$ (6.7)

Proof. We show that $(i) \Longrightarrow (ii) \Longrightarrow (iii) \Longrightarrow (\mathrm{i}v) \Longrightarrow (v) \Longrightarrow (vi) \Longrightarrow (i)$.

$(i) \Longrightarrow (ii)$. Let $\beta = f(x) \leq f(y) \ \Longrightarrow\ x, y \in L(\beta)$.

Since $L(\beta)$ is a convex set $\lambda x + (1-\lambda)y \in L(\beta)$. By the definition of $L(\beta)$

$$f(\lambda x + (1-\lambda)y) \geq \beta = f(x).$$

$(ii) \Longrightarrow (iii)$. We show that not (*iii*) implies not (*ii*). Not (*iii*) means that there exist $x, y \in I,\ x \neq y$ such that

$$f'(x)(y-x) < 0 \tag{6.8}$$

and

$$f(y) \geq f(x). \tag{6.9}$$

Define the function of one variable h for $0 \leq \alpha \leq 1$ by

$$h(\alpha) = f\big(x + \alpha(y - x)\big). \tag{6.10}$$

It can be verified that

$$h(0) = f(x) \quad and \;\; g(1) = f(y). \tag{6.11}$$

It can also be verified that the derivative of $h(\alpha)$ for $0 \leq \alpha \leq 1$ can be computed as follows

$$h'(\alpha) = f'\big(x + \alpha(y - x)\big)(y - x). \tag{6.12}$$

Evaluating (6.12) at $\alpha = 0$ and using (6.8) shows that

$$h'(0) < 0. \tag{6.13}$$

Since the first order partial derivative of f is continuous, it can be seen that (6.13) implies the existence of a ε such that

$$0 < \varepsilon < 1 \tag{6.14}$$

and

$$h'(\alpha) < 0 \tag{6.15}$$

for all α such that $0 \leq \alpha \leq \varepsilon$.

Thus $h(\alpha)$ is a decreasing function over this interval of α's and thus

$$g(\varepsilon) = f\big(x + \varepsilon(y - x)\big) = f((1 - \varepsilon)x + \varepsilon y)) < h(0) = f(x). \tag{6.16}$$

But (6.14) and (6.16) imply that

$$f(\varepsilon(x + (1 - \varepsilon)y) < f(x) \tag{6.17}$$

where $\lambda \equiv 1 - \varepsilon$. Since (6.14) implies $0 < \lambda < 1$, (6.17) contradicts (ii).

$(iii) \Longrightarrow (iv)$. Since I is open, there exists $\varepsilon > 0$ such that when $|\theta| < \varepsilon, x + \theta(y - x) \in I$.

Since f is twice differentiable at x, by Taylor's Theorem, we have

$$f\big(x + \theta(y - x)\big) = f(x) + \theta(y - x)f'(x) + \tfrac{1}{2}\theta^2(y - x)^2 f''(x) \tag{6.18}$$

Observe that if

$$f'(x)(y - x) < 0 \tag{6.19}$$

then

$$\theta f'(x)(y-x) < 0 \tag{6.20}$$

Now, since $f'(x)(y-x) < 0 \quad \Rightarrow \quad f(y) < f(x)$ it follows that

$$\theta f'(x)(y-x) + f(y) < f(x). \tag{6.21}$$

From (6.18) and (6.21) we have

$$2\theta f'(x)(y-x) + f(y) + \tfrac{1}{2}\theta^2(y-x)^2 f''(x) < f\big(x+\theta(y-x)\big) \tag{6.22}$$

Since $f'(x)(y-x) = 0$, (6.22) become

$$\Rightarrow \tfrac{1}{2}\big(\theta(y-x)\big)^2 f''(x) < f\big(y+\theta(y-x)\big) \tag{6.23}$$

Dividing through by θ^2 and letting $\theta \longrightarrow 0$, we have that

$$(y-x)^2 f''(x) < 0$$

Furthermore, since $f(y) < f(x)$ means that f attains a minimum at y with $x \neq y$ it follows that $x - y \neq 0$ so that $f'(x)(y-x) < 0 \Rightarrow f'(x) \neq 0$.

Thus

$$h(\alpha) = f(x + \alpha(y-x))$$

does not attain a semistrict local minimum at $\alpha = 0$.

$(iv) \Rightarrow (v)$. It is sufficient to show that (v) is equivalent to the statement

$$h(\alpha) \equiv f\big(w + \alpha(y-x)\big), \quad w \in I, \;\; 0 \neq y - x \in \mathbb{R} \tag{6.24}$$

does not attain a semistrict local minimum at $\alpha = 0$.

Suppose (iv) occurs, then $h(\alpha)$ attains a strict local minimum at $\alpha = 0$ and hence cannot attain a semistrict local minimum at $\alpha = 0$.

$(v) \Rightarrow (vi)$. This is equivalent to showing that not (vi) implies not (v). Suppose f is not quasiconcave, then there exist $x, y \in I$ and α^* such that $\lambda^* \in (0,1)$ and

$$f(\lambda^* x_1 + (1-\lambda^*)x_2) < \min\,\{f(x), f(y)\}. \tag{6.25}$$

Define $h(t) \equiv f(x + \alpha(y - x)$ for $0 \leq \alpha \leq 1$. Since f is assumed to satisfy the line segment minimum property there exists a α^* such that $0 \leq \alpha \leq 1$ and

$$h(\alpha^*) = \min_{\alpha} \ \{h(\alpha): \ 0 \leq \alpha \leq 1\}. \tag{6.26}$$

The definition of h and (6.25) show that α^* satisfies $0 < \alpha^* < 1$ and

$$f\big(x + \alpha^*(y - x)\big) = f\big((1 - \alpha^*)x + \alpha^* y\big) < \min \ \{f(x), f(y)\}. \tag{6.27}$$

Thus f attains a semistric local minimum, which contradicts (v).

$(vi) \Longrightarrow (i)$. Let $\beta \in Range\, f, \ x, y \in L(\beta)$ and $\lambda \in (0,1)$,

$$\Longrightarrow \quad f(x) \geq \beta \text{ and } f(y) \geq \beta. \tag{6.28}$$

From (*vi*), we have that

$$f(\lambda x + (1 - \lambda)y) \geq min \ \ \{f(x), \ f(y)\} \geq \beta. \tag{6.29}$$

where the last inequality follows by using (6.28). But (6.29) shows that
$\lambda x + (1 - \lambda)y \in L(\beta)$ and thus $L(\beta)$ is a convex set. ■

6. Concluding Remarks

Although this characterization does not incorporate non-differentiable quasiconcave functions, it shows that differentiable quasiconcave functions which satisfy any of the defining properties stated in equations (4.1) and (4.4) to (4.9) equally satisfy all the other properties stated in equations (6.1) to (6.7) in the result above.

The result further shows that if a given definition of quasiconcavity cannot be incorporated into a given scheme one can resort to another, thereby providing safe havens for a number of computational schemes. On the other hand, stemming from the interplay among these concepts, a given scheme can be refined to incorporate a desired definition or concept.

As stated earlier some of these defining properties of quasiconcave functions do exist in optimization materials, however, to the best of my knowledge, this characterization which combines these properties, thereby giving a wider definition of quasiconcavity has not been achieved before now. Therefore this work puts us at a better horizon for recognizing quasiconcave functions.

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