SYMMETRIC QUANTUM MARKOVIAN SEMIGROUPS

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ABSTRACT

Using symmetric embeddings of the von Neumann algebra $\mathcal{M}$ into Trunov non-commutative L_p–spaces, we study symmetric quantum Markovian semigroups. We obtained a characterization of the generator as a Dirichlet form for such symmetric Markovian semigroups.

Keywords: von Neumann algebras, non commutative L_p –spaces, quantum Markov semigroup, density matrix h^{α} , Dirichlet form.

1. Introduction:

Inspired by the work of Goldstein and Lindsay [1] on K.M.S-symmetric semigroups on von Neumann algebras, we study symmetric Markov semigroups within the context of Trunov's L_p- spaces defined over a von Neumann algebra $\mathcal{M}$. The construction of a tracially symmetric markov semigroups in the setting of a semi-finite von Neumann algebra $\mathcal{M}$ admitting a faithful normal semi-finite trace τ on $\mathcal{M}$ was initiated by Albeverio and Høegh-Krohn [2] in the seventies and developed by Davies and Lindsay [3] in the nineties. Their construction and analysis took place on the Segal $L_p(\tau)$ spaces which together with each of the interpolating spaces $L_p(\tau)$ $(0 \leq p \leq \infty)$, is a subspace of a topological *-algebra $\mathcal{M}^{\sim}$ of τ- measurable operators acting as closed densely defined operators on $L_p(\tau)$. Goldstein and Lindsay [1] extends it to the context of state φ on von Neumann algebras in the nineties, by the use of symmetric embedding of the von Neumann algebra into the Haagerup spaces L_p- spaces. In this paper we study symmetric markov semigroups using the embedding defined by them, on the algebra $\mathcal{M}$ into the Trunov's space $L_p(\mathcal{M})$ [4].

2. PRELIMINARIES:

A von Neumann algebra is a *-subalgebra $\mathcal{M}$ of $\mathcal{B}(\mathfrak{H})$ which is self-adjoint, contains the identity operator I and is closed in the weak operator topology. The weak operator topology is induced by the family of semi norms $\{p_{\xi,\eta}\}$ defined on $\mathcal{M}$ by $p_{\xi,\eta}(x) = \sum|\langle x\xi, \eta\rangle|$, with $x \in \mathcal{M}, \xi, \eta \in \mathfrak{H}$. $\mathcal{M}_+$ denote the positive elements of $\mathcal{M}$, i.e $\mathcal{M}_+ = \{x \in \mathcal{M}: x \geq 0\}$. A linear positive functional φ on $\mathcal{M}$ is called a state if $\varphi(1) = 1$. The space of all σ-weakly continuous linear functionals on a von Neumann algebra $\mathcal{M}$ is called the predual $\mathcal{M}_*$, we denote by $\mathcal{M}_{*,+}$ the positive part of $\mathcal{M}_*$. More details on von Neumann algebras will be found in [5,6,7]

We recall the Trunov construction of the L_p- spaces as follows; Let $\mathcal{M}$ be semi-finite von Neumann algebra, with a faithful normal state φ, there exist a unique operator $\rho \in L_1^+(\mathcal{M})$ called the Radon-Nikodym derivative of the state with respect to the trace τ such that

$$\varphi(x) = \tau(x.\rho) = \tau(\rho.x), \qquad [8]$$

This representation enables one to define for each number $1 \leq p \leq \infty$ a certain norm on $\mathcal{M}$ that is connected with φ. For any $x \in \mathcal{M}$, the operator $\rho^{1/2p}x\rho^{1/2p} \in L_p(\mathcal{M})$, and therefore the following definition make sense,

$$\|x\|_p = (\tau|\rho^{1/2p}.x.\rho^{1/2p}|^p)^{1/p}$$

This defines a norm on $\mathcal{M}$ for each, $1 \leq p \leq \infty$, and we write $\|x\|_\infty = \|x\|$, for $x \in \mathcal{M}$. This norm $\|x\|_p$ does not depend on the choice of the faithful normal semi-finite trace τ and is a norm on $\mathcal{M}$ [6].

Let $L_p(\mathcal{M}) = \{x \in \mathcal{M} : \|x\|_p < \infty\}$ be the completion of $\mathcal{M}$ with respect to this norm, this completion we called the Trunov L_p −spaces,with $\mathcal{M} = L_\infty(\mathcal{M})$ in the usual norm $\|.\|_\infty$.

As for the *density matrix*, using functional calculus, we introduce the operator h^α for $\alpha \in \mathbb{R}_+$ as follows: Let h be a positive non-singular self-adjoint operator with spectrum $sp(h)$. We denote by $C(sp(h))$ the von Neumann algebra of continuous real valued functions on $sp(h)$. Let $f_\alpha \in C(sp(h))$ be defined by

$$f_\alpha(s) = s^\alpha, \qquad s \in sp(h) . \qquad [9]$$

Then h^α is define as $f_\alpha(h)$ for real values of α. Hence we have $f_\alpha(h) = h^\alpha$. Now for a von Neumann algebra $\mathcal{M}$ with a faithful normal trace τ. We assume that h^α satisfies the following three conditions;

(i) $\tau(h^\alpha) < \infty$,

(ii) $\tau(h^\alpha) = 1$,

(iii) $\varphi(x) = \tau(x.h^\alpha) = \tau\left(h^{\frac{\alpha}{2}}.x.h^{\frac{\alpha}{2}}\right)$, where is a faithful normal state φ on $\mathcal{M}$.

With the preceding conditions, h^{α} is a trace class operator called the density matrix or the Radon-Nikodym derivative of the trace τ with respect to the state φ , that is , $\frac{\partial\tau}{\partial\varphi} = h^{\alpha}$, hence $h^{\alpha} \in L_1(\mathcal{M})_+$ - the set of trace class operators.

Now for $1 \le p \le q < \infty$, let $p = \frac{1}{1-t}$ and $q = \frac{1}{t}$, with $0 < t < 1$ then $\frac{1}{p} + \frac{1}{q} = 1$. Since $h^{\alpha} \in L_1(\mathcal{M})_+$, we have $h^{\frac{\alpha}{2}} \in L_2(\mathcal{M})$, and hence $h^{(1-t)\alpha} \in L_p(\mathcal{M})$.

We now have the following definitions of technical terms :

Definition 1: symmetric embeddings of

(i) $\mathcal{M}$ into $\mathcal{M}_*$, by $\quad i(x) = \tau\left(h^{\frac{\alpha}{2}}.x.h^{\frac{\alpha}{2}}\right), \quad x \in \mathcal{M}.$

(ii) $\mathcal{M}$ into $L_p(\mathcal{M})$, by $\quad i^p(x) = \left(h^{\frac{(1-t)\alpha}{2}}.x.h^{\frac{(1-t)\alpha}{2}}\right), \quad x \in \mathcal{M}.$

(iii) $L_p(\mathcal{M})$ into $L_1(\mathcal{M})$, by $\quad \kappa(f) = \left(h^{\frac{\alpha t}{2}}.f.h^{\frac{\alpha t}{2}}\right) \quad f \in L_p(\mathcal{M}).$

Where $L_p(\mathcal{M})$, is the Trunov's $L_P -$ space.

Definition 2: A linear operator $T: \mathcal{M} \to \mathcal{M}$ is called Markov if $0 \le x \le 1$ implies that $0 \le Tx \le 1$.

Definition 3: An operator T on L_p is called L_p-Markov if $0 \le f \le h^{(1-t)\alpha}$ implies that $0 \le Tf \le h^{(1-t)\alpha}, \quad f \in L_p$.

Definition 4: A linear operator $T: \mathcal{M} \to \mathcal{M}$ is called Symmetric or tracially Symmetric, if

$$\tau\left(h^{\frac{\alpha}{2}}.Tx.h^{\frac{\alpha}{2}}\,y\right) = \tau\left(x\,h^{\frac{\alpha}{2}}.Ty.h^{\frac{\alpha}{2}}\right), \qquad x, y \in \mathcal{M}.$$

Definition 5: A Markov semigroup P_t on $\mathcal{M}$ is a weak* -continuous semigroup consisting of Markov operators.

Definition 6: An L_P −Markov semigroup $(P_t)_{t\geq 0}$, is a strongly continuous contraction semigroup consisting of L_P markov operators.

Definition 7: A Markov Semigroup $(P_t)_{t\geq 0}$ is called Symmetric Markov Semigroup, if

$$\tau\left(h^{\frac{\alpha}{2}}.P_t x.h^{\frac{\alpha}{2}}\ y\right) = \tau\left(x\ h^{\frac{\alpha}{2}}.P_t y.h^{\frac{\alpha}{2}}\right) \qquad \text{for} \quad x, y \in \mathcal{M}.$$

Definition 8: An $L_P -$ Markov resolvent family $(R_\lambda)_{\lambda\geq 0}$ is a strongly continuous contraction resolvent such that each λR_λ is $L_P -$ Markov.

Definition 9: By a strongly continuous contraction resolvent we mean a family $(R_\lambda)_{\lambda>0}$ of (everywhere defined) linear operators on $\mathfrak{H}$ satisfying the following conditions,

(i) $\lim_{\lambda\to\infty} \lambda R_\lambda x = x \quad for \quad x \in \mathfrak{H}$

(ii) λR_λ is a contraction on $\mathfrak{H}$ for all $\lambda > 0$

(iii) $R_\lambda - R_\mu = (\mu - \lambda)R_\lambda R_\mu,$ for all $\lambda, \mu > 0$.

Definition 10: A closed densely defined operator G *on* $L_2(\mathcal{M})$ is called Dirichlet if G is real ,that is the domain of G is $*$ -invariant and $Gx^* = Gx$ for $x \in D(G)$ and $\langle Gx, \left(x - h^{\frac{\alpha}{2}}\right)_+ \rangle \leq 0$ for all $D_h(G)$. Where $D(G)$is the domain of G and $D_h(G)$ is the domain of the self adjoint part of G.

Definition 11: With any $x \in L_2(\mathcal{M})_h$, the self adjoint part of $L_2(\mathcal{M})$, we associate an element x_e defined in [10] by the formula,

$$x_e = x - \left(x - h^{\frac{\alpha}{2}}\right)_+ = h^{\frac{\alpha}{2}} - \left(x - h^{\frac{\alpha}{2}}\right)_-$$

we have that $x_e \leq x$, $x_e \leq h^{\frac{\alpha}{2}}$ and $h^{\frac{\alpha}{2}} \in L_2(\mathcal{M})$.

Definition 12: A non negative quadratic form $\mathcal{E}$ defined on a dense subspace $D(\mathcal{E})$ of a Hilbert space $\mathfrak{H}$ is closed if the domain $D(\mathcal{E})$ equipped with the norm $\|.\|_{\mathcal{E}}$ given by $\|x\|_{\mathcal{E}}^2 = \|x\|^2 + \mathcal{E}(x)$ is a Hilbert space. It is closable if there exists a closed form $\hat{\mathcal{E}}$ extending $\mathcal{E}$, that is, $D(\hat{\mathcal{E}}) \supset D(\mathcal{E})$ and $\hat{\mathcal{E}}(x) = \mathcal{E}(x)$ for $x \in D(\mathcal{E})$. There exists in such a case a closed form $\overline{\mathcal{E}}$, called the closure of $\mathcal{E}$,which is the smallest closed extension of $\mathcal{E}$.

Definition 13: A nonnegative quadratic form $\mathcal{E}$ on $L_2(\mathcal{M})$ with dense domain $D(\mathcal{E})$ is called Dirichlet if,

(i) $\mathcal{E}$ is real ,that is $D(\mathcal{E})$ is * -invariant and $\mathcal{E}x^* = \mathcal{E}x$ for $x \in D(\mathcal{E})$.

(ii) $x_{+}, x_{\mathrm{e}} \in D_h(\mathcal{E})$ for $x \in D_h(\mathcal{E})$,

(iii) $\mathcal{E}x_{+} \leq \mathcal{E}x$ and $\mathcal{E}x_{\mathrm{e}} \leq \mathcal{E}x$ $for\ x \in D_h(\mathcal{E})$.

where $D_h(\mathcal{E})$ is the self-adjoint part of $D(\mathcal{E})$.

3. RESULTS:

The main result in this section is theorem 2, which state, that the form generator of a symmetric Markov semigroup is a Dirichlet form. The converse of the theorem was obtained for tracially symmetric Markov semigroups acting on Segal L_2-spaces by Albeverio and Høegh-Krohn [2]. We are now ready to embark on the proof of results ;

Theorem: 1

Let $(R_\lambda)_{\lambda>0}$ be a strongly continuous contraction resolvent on $L_2(\mathcal{M})$ and G the generator of the semigroup $(P_t)_{t\geq 0}$ .Then the following conditions are equivalent

(i) each λR_λ is L_2 –Markov

(ii) each P_t is L_2 –Markov

(iii) G is Dirichlet

Proof:

$(i) \Rightarrow (ii)$

This follows from the relation $P_t = s.t\ \lim_{n\to\infty}(n/tR_{n/t})^n$

$(ii) \Rightarrow (iii)$

Since P_t an L_2 –Markov contraction, then for $x \in L_2(\varphi)_h$

we have $x = \left(x - h^{\frac{\alpha}{2}}\right)_+ + x - \left(x - h^{\frac{\alpha}{2}}\right)_+$

$$x_{\mathrm{e}} = x - \left(x - h^{\frac{\alpha}{2}}\right)_+ = h^{\frac{\alpha}{2}} - \left(x - h^{\frac{\alpha}{2}}\right)_-$$

and also, $\langle\left(x - h^{\frac{\alpha}{2}}\right)_+, \left(x - h^{\frac{\alpha}{2}}\right)_-\rangle = 0.$

Hence,

$$\langle P_t x, \left(x - h^{\frac{\alpha}{2}}\right)_+ \rangle = \langle P_t \left(x - h^{\frac{\alpha}{2}}\right)_+, \left(x - h^{\frac{\alpha}{2}}\right)_+ \rangle + \langle P_t x - \left(x - h^{\frac{\alpha}{2}}\right)_+, \left(x - h^{\frac{\alpha}{2}}\right)_+ \rangle$$

since $x_{\mathrm{e}} = x - \left(x - h^{\frac{\alpha}{2}}\right)_+$

we have $\langle P_t x, \left(x - h^{\frac{\alpha}{2}}\right)_+ \rangle = \langle P_t \left(x - h^{\frac{\alpha}{2}}\right)_+, \left(x - h^{\frac{\alpha}{2}}\right)_+\rangle + \langle P_t x_{\mathrm{e}} \left(x - h^{\frac{\alpha}{2}}\right)_+ \rangle$

and $x_{\mathrm{e}} \leq h^{\frac{\alpha}{2}}$, hence we have,

$$\langle P_t x, \left(x - h^{\frac{\alpha}{2}}\right)_+ \rangle \leq \langle P_t \left(x - h^{\frac{\alpha}{2}}\right)_+, \left(x - h^{\frac{\alpha}{2}}\right)_+\rangle + \langle P_t h^{\frac{\alpha}{2}}, \left(x - h^{\frac{\alpha}{2}}\right)_+ \rangle$$

using the property of contractivity of the semigroup on the right, we have

$$\langle P_t x, \left(x - h^{\frac{\alpha}{2}}\right)_+ \rangle \leq \langle\left(x - h^{\frac{\alpha}{2}}\right)_+, \left(x - h^{\frac{\alpha}{2}}\right)_+\rangle + \langle h^{\frac{\alpha}{2}}, \left(x - h^{\frac{\alpha}{2}}\right)_+ \rangle$$

from the relation $x - h^{\frac{\alpha}{2}} = \left(x - h^{\frac{\alpha}{2}}\right)_+ - \left(x - h^{\frac{\alpha}{2}}\right)_-$.

We have,

$$\langle P_t x, \left(x - h^{\frac{\alpha}{2}}\right)_+ \rangle \leq \langle x - h^{\frac{\alpha}{2}}, \left(x - h^{\frac{\alpha}{2}}\right)_+ \rangle + \langle h^{\frac{\alpha}{2}}, \left(x - h^{\frac{\alpha}{2}}\right)_+ \rangle$$

$$\langle P_t x, \left(x - h^{\frac{\alpha}{2}}\right)_+ \rangle \leq \langle x - h^{\frac{\alpha}{2}} + h^{\frac{\alpha}{2}}, \left(x - h^{\frac{\alpha}{2}}\right)_+ \rangle$$

i.e $$\langle P_t x, \left(x - h^{\frac{\alpha}{2}}\right)_+ \rangle \leq \langle x, \left(x - h^{\frac{\alpha}{2}}\right)_+ \rangle$$

since P_t is a contraction and $x_{\mathrm{e}} \leq h^{\frac{\alpha}{2}}$,

then, for all $x \in D_h(G)$ we have, $\langle P_t x - x, \left(x - h^{\frac{\alpha}{2}}\right)_+ \rangle \leq 0$

which implies that

$$\langle Gx, \left(x - h^{\frac{\alpha}{2}}\right)_+ \rangle = \lim_{t \downarrow 0} \frac{1}{t} \langle P_t x - x, \left(x - h^{\frac{\alpha}{2}}\right)_+ \rangle \leq 0$$

$$\langle Gx, \left(x - h^{\frac{\alpha}{2}}\right)_+ \rangle = \lim_{t \downarrow 0} \frac{1}{t} \langle (P_t - I)x, \left(x - h^{\frac{\alpha}{2}}\right)_+ \rangle \leq 0$$

$(iii) \Rightarrow (i)$

Let $x \in L_2(\varphi)_h$ and $y = \lambda R_\lambda x$, if $x \leq h^{\frac{\alpha}{2}}$, then

$$\lambda \langle y, \left(y - h^{\frac{\alpha}{2}}\right)_+ \rangle = \langle \lambda y - Gy, \left(y + h^{\frac{\alpha}{2}}\right)_+ \rangle + \langle Gy, \left(x - h^{\frac{\alpha}{2}}\right)_+ \rangle \leq \lambda \langle x, \left(y - h^{\frac{\alpha}{2}}\right)_+ \rangle$$

since $\lim_{\lambda \to \infty} \lambda R_\lambda x = x$

$$\lambda \langle y, \left(y - h^{\frac{\alpha}{2}}\right)_+ \rangle \leq \lambda \langle h_\varphi^{\frac{(1-t)}{2}}, \left(y - h^{\frac{\alpha}{2}}\right)_+ \rangle$$

for this inequality to hold we must have that $y \leq h_\varphi^{\frac{(1-t)}{2}}$

hence we have,

$$\lambda\langle y,\left(y-h^{\frac{\alpha}{2}}\right)_+\rangle-\lambda\langle h^{\frac{\alpha}{2}},\left(y-h^{\frac{\alpha}{2}}\right)_+\rangle \leq 0$$

$$\lambda\langle\left(y-h^{\frac{\alpha}{2}}\right),\left(y-h^{\frac{\alpha}{2}}\right)_+\rangle \leq 0$$

$$\lambda\langle y-h^{\frac{\alpha}{2}},\left(y-h^{\frac{\alpha}{2}}\right)_+\rangle=\lambda\langle\left(y-h^{\frac{\alpha}{2}}\right)_+,\left(y-h^{\frac{\alpha}{2}}\right)_+\rangle-\lambda\langle\left(y-h^{\frac{\alpha}{2}}\right)_-,\left(y-h^{\frac{\alpha}{2}}\right)_+\rangle \leq 0$$

hence $\lambda\langle\left(y-h^{\frac{\alpha}{2}}\right)_-,\left(y-h^{\frac{\alpha}{2}}\right)_+\rangle=0$

thus we have,

$\lambda\langle\left(y-h^{\frac{\alpha}{2}}\right)_+,\left(y-h^{\frac{\alpha}{2}}\right)_+\rangle \leq 0$ this implies that $\lambda\left\|\left(y-h^{\frac{\alpha}{2}}\right)_+\right\| \leq 0$

since $\lambda>0$, we have $\left\|\left(y-h^{\frac{\alpha}{2}}\right)_+\right\| \leq 0$, this implies that $0 \leq \lambda R_\lambda x \leq h^{\frac{\alpha}{2}}$

and if $x \geq 0$, then $-\mathrm{n}x \leq h^{\frac{\alpha}{2}}$, which implies $-ny \leq h^{\frac{\alpha}{2}}$ for all $n \in \mathbb{N}$, hence $y \geq 0$.

Theorem: 2

Let P be a symmetric $L_2(\mathcal{M})$- Markov operator. Then the quadratic form

$x \in L_2(\mathcal{M}) \longmapsto \mathcal{E}x=\langle(I-P)x,x\rangle \in L_2(\mathcal{M})$ is a Dirichlet form.

Proof:

By assumption, P is a self adjoint, positivity- preserving contraction and $\mathcal{E}$ is a non negative, real quadratic form on $L_2(\mathcal{M})$. We have for $x \in L_2(\mathcal{M})$ and P self adjoint,

$$\begin{aligned}\langle Px,x\rangle&=\langle P(x_+-x_-),(x_+-x_-)\rangle\\&=\langle Px_+,(x_+-x_-),\rangle-\langle Px_-,(x_+-x_-)\rangle\\&=\langle Px_+,x_+\rangle-\langle Px_+,x_-\rangle-\langle Px_-,x_+\rangle+\langle Px_-,x_-\rangle\\&=\langle Px_+,x_+\rangle+\langle \mathrm{P}x_-,x_-\rangle-2\langle Px_-,x_+\rangle\end{aligned}$$

therefore

$$\langle (I-P)x, x\rangle = \langle (I-P)x_+ , x_+\rangle + \langle (I-P)x_- , x_-\rangle - 2\langle (I-P)x_- , x_+\rangle$$

$$\mathcal{E}x = \langle (I-P)x, x\rangle = \mathcal{E}x_+ + \mathcal{E}x_- - 2\langle (I-P)x_- , x_+\rangle$$

$$\mathcal{E}x - \mathcal{E}x_+ = \mathcal{E}x_- - 2\langle (I-P)x_-, x_+ \rangle \geq 0 \ .$$

now we have,

$$\mathcal{E}x - \mathcal{E}x_+ = \mathcal{E}x_- - 2\langle (I-P)(x_+ - x),\ x_+ \rangle$$

$$\mathcal{E}x - \mathcal{E}x_+ = \mathcal{E}x_- - 2\langle (I-P)x_+,\ x_+ \rangle + 2\langle (I-P)x,\ x_+ \rangle$$

$$\mathcal{E}x = \mathcal{E}x_+ + \mathcal{E}x_- - 2\langle (I-P)x_+,\ x_+ \rangle + 2\langle (I-P)x,\ x_+ \rangle$$

now since $x = x_{\mathrm{e}} + \left(x - h^{\frac{\alpha}{2}}\right)_+$, $\mathcal{E}|x| = \mathcal{E}x_+ + \mathcal{E}x_-$ hence,

$$\mathcal{E}x_{\mathrm{e}} = \mathcal{E}|x| - \mathcal{E}\left(x - h^{\frac{\alpha}{2}}\right)_+ - 2\left\langle (I-P)\left(x - h^{\frac{\alpha}{2}}\right)_+ ,\ \left(x - h^{\frac{\alpha}{2}}\right)_+ \right\rangle$$

$$+2\left\langle (I-P)\left(x - h^{\frac{\alpha}{2}}\right)_+ - \left(x - h^{\frac{\alpha}{2}}\right)_- + h^{\frac{\alpha}{2}} ,\ \left(x - h^{\frac{\alpha}{2}}\right)_+ \right\rangle$$

$$\mathcal{E}x_{\mathrm{e}} = \mathcal{E}|x| - \mathcal{E}\left(x - h^{\frac{\alpha}{2}}\right)_+ - 2\left\langle (I-P)\left(x - h^{\frac{\alpha}{2}}\right)_- + h^{\frac{\alpha}{2}},\ \left(x - h^{\frac{\alpha}{2}}\right)_+ \right\rangle$$

$$\mathcal{E}x_{\mathrm{e}} = \mathcal{E}|x| - \mathcal{E}\left(x - h^{\frac{\alpha}{2}}\right)_+ - 2\left\langle (P-I)\left(\left(x - h^{\frac{\alpha}{2}}\right)_- - h^{\frac{\alpha}{2}}\right),\ \left(x - h^{\frac{\alpha}{2}}\right)_+ \right\rangle$$

$$\mathcal{E}x_{\mathrm{e}} = \mathcal{E}|x| - \mathcal{E}\left(x - h^{\frac{\alpha}{2}}\right)_+ - 2\left\langle P\left(x - h^{\frac{\alpha}{2}}\right)_-, \left(x - h^{\frac{\alpha}{2}}\right)_+ \right\rangle - 2\left\langle (I-P)h^{\frac{\alpha}{2}}, \left(x - h^{\frac{\alpha}{2}}\right)_+ \right\rangle$$

since $\mathcal{E}|x| \leq \mathcal{E}\,x$ we have,

$$\mathcal{E}x_{\mathrm{e}} \leq \mathcal{E}x - \mathcal{E}\left(x - h^{\frac{\alpha}{2}}\right)_+ - 2\left\langle P\left(x - h^{\frac{\alpha}{2}}\right)_-, \left(x - h^{\frac{\alpha}{2}}\right)_+ \right\rangle - 2\left\langle (I-P)h^{\frac{\alpha}{2}}, \left(x - h^{\frac{\alpha}{2}}\right)_+ \right\rangle.$$

Since P is an $L_2(\mathcal{M})$- Markov operator from definition we have $Ph^{\frac{\alpha}{2}} \leq h^{\frac{\alpha}{2}}$ for all $h^{\frac{\alpha}{2}} \in L_2(\mathcal{M})$. Thus $(I-P)h^{\frac{\alpha}{2}} \geq 0$ hence the last term $\left\langle (I-P)h^{\frac{\alpha}{2}}, \left(x - h^{\frac{\alpha}{2}}\right)_+ \right\rangle \geq 0$ likewise the other terms are non-negative since they are inner product of non-negative operators.

Hence we have

$$\mathcal{E}x_e + \mathcal{E}\left(x - h^{\frac{\alpha}{2}}\right)_+ + 2\langle P\left(x - h^{\frac{\alpha}{2}}\right)_-, \left(x - h^{\frac{\alpha}{2}}\right)_+\rangle + 2\langle (I-P)h^{\frac{\alpha}{2}}, \left(x - h^{\frac{\alpha}{2}}\right)_+\rangle \leq \mathcal{E}x$$

this implies that $\mathcal{E}x_e \leq \mathcal{E}x$ for $x \in D_h(\mathcal{E})$.

Hence the quadratic form $\mathcal{E}(x) = \langle (I-P)x, x\rangle$ is Dirichlet.

Conclusion: Symmetric Markov semigroups and the Dirichlet forms on noncommutative spaces plays an important role in noncommutative potential theory, for example the Dirichlet energy integral $\mathcal{E}[u]$ can be use to describe potential at the vertexes of a electrical circuit [11]. The Dirichlet forms from the dynamical point of view is also fundamental, since a Dirichlet form $\mathcal{E}$ acting on a locally compact topological Hausdorff space gives rise to a family of Markov-Hunt stochastic processes.

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