Journal of the Nigerian Association of Mathematical Physics
Volume **20 (March, 2012), pp 505 – 510**

On Certain Sufficient Conditions For A Function To Be Close-To-Convex Function

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Abstract

We consider certain properties of $1-\left(z\,\dfrac{g'(z)}{f'(z)}\right)'$ ***as a sufficient condition for a function*** $f(z)$ ***to be a close-to-convex function.***

Keywords: Univalent function, Convex function, Starlike function, Close-to-convex function.
2000 Mathematics Subject Classification. Primary 30C45.

1.0 Introduction

Let A denote the class of functions $f(z)$ which are analytic in the unit disc $U=\{z:|z|<1\}$ with $f(0)=f'(0)-1=0$. For a function $f(z)\in A$, we say that it is starlike [1] in the unit disc *U* if and only if

$$\operatorname{Re}\left\{\frac{z\,f'(z)}{f(z)}\right\}>0$$

for all $z\in U$. We denote by *S** the class of all such functions.

We denote by K the class of convex functions [1] in the unit disc *U*, i.e. the class of univalent functions $f(z)\in A$ for which

$$\operatorname{Re}\left\{1+\frac{z\,f''(z)}{f'(z)}\right\}>0$$

for all $z\in U$.

A function $\dot{f}(z)$ regular in the unit disk $|z|<1$ is said to be close-to-convex function [1] if there is a convex function *g(z)* such that

$$\operatorname{Re}\left[\frac{f'(z)}{g'(z)}\right]>0$$

It is clearly known [2] that if $\operatorname{Re}\left[\dfrac{f'(z)}{g'(z)}\right]>0$ for $|z|<1$, then $f(z)$ is close-to-convex. Denote the class of close-to-convex functions by C*.

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The above mentioned classes are subclasses of univalent functions in *U* and more $K \subset S^* \subset C^*$.

Let $f(z)$ and $\mu(z)$ be analytic in the unit disc. Then we say that $f(z)$ is subordinate to $\mu(z)$, and we write $f(z) \prec \mu(z)$, if $\mu(z)$ is univalent in *U*, *f(0)* = $\mu(0)$ and $f(U) \subseteq \mu(U)$.

In this paper, we use the method of differential subordinations. The general theory of differential subordinations introduced by Miller and Mocanu is given in [3]. Namely, if $\phi : C^2 \to C$ (where *C* is the complex plane) is analytic in domain $D,\ and\ h(z)$ is univalent in *U*, and if $p(z)$ is analytic in *U* with $(p(z), zp'(z)) \in D\ when\ z \in U$, then we say that $p(z)$ satisfies a first – order differential subordination if

$$\phi(p(z), zp'(z)) \prec h(z) \tag{1.1}$$

We say that the univalent function $\mu(z)$ is dominant of the differential subordination (1.1) if $p(z) \prec q(z)$ for all $p(z)$ satisfying (1.1). If $\tilde{q}(z)$ is a dominant of (1.1) and $\tilde{q}(z) \prec q(z)$ for all dominants of (1.1), then we say that $\tilde{q}(z)$ is the best dominant of the differential subordination (1.1).

In the following section, we need the following lemma of Miller and Mocanu [4].

1.1 Lemma [4]

Let $q(z)$ be univalent in the unit disc *U*, and let $\theta(w)$ and ϕ (w) be analytic in a domain *D* containing $q(U)$, with $\phi(w) \neq 0\ when\ w \in q(U)$. Set $Q(z)\, zq'(z)\, \phi(q(z)),\ h(z) = \theta(q(z)) + Q(z)$, and suppose that

a. $Q(z)$ is starlike in the unit disc *U*,

b. $\operatorname{Re}\left\{\dfrac{zh'(z)}{Q(z)}\right\} = \operatorname{Re}\left\{\dfrac{\theta'(q(z))}{\phi(q(z))} + \dfrac{zQ'(z)}{Q(z)}\right\} > 0,\ z \in U$.

If $p(z)$ is analytic in *U*, with $p(0) = q(0),\ p(U) \subseteq D$ and

$$\theta(p(z)) + zp'(z)\,\phi(p(z)) \prec \theta(q(z) + zq'(z)\,\phi(q(z))) = h(z) \tag{1.2}$$

then $p(z) \prec q(z)$ and $p(z)$ is the best dominant of (1.2). Even more we need the following lemma, which in more general form is due to Hallenbeck and Ruscheweyh [5].

1.2 lemma [5]

Let $G(z)$ be a convex univalent in $U, G(0) = 1$. Let $F(z)$ be analytic in $U,\ F(0) = 1$ and let $F(z) \prec G(z)$ in *U*. Then for all $n \in N_0$

$$(n+1)z^{-n-1} \int_0^z t^n F(t)\,dt \prec (n+1)z^{-n-1} \int_9^z t^n G(t)\,dt \tag{1.3}$$

2.0 Main Result and Consequences

We shall make use of lemma 1.1 and 1.2 to obtain some conditions for $1 - \left(z\,\dfrac{g'(z)}{f'(z)}\right)'$ which leads to close-to-convex function.

2.1 Theorem

If $f, g \in A$ and

$$1 - \left(z\,\frac{g'(z)}{f'(z)}\right)' \prec 2 - \frac{2}{(1+z)^2} = h(z) \tag{2.1}$$

then $f \in C^*$.

Proof.

We choose $p(z) = \dfrac{f'(z)}{g'(z)};\ q(z) = \dfrac{1+z}{1-z};\ \phi(w) = \dfrac{1}{w^2};\ \theta(w) = 1 - \dfrac{1}{w}$. Then $q(z)$ is univalent in $U;\ \theta(w)\ and\ \phi(w)$ are analytic with domain $D = C\backslash\{0\}$ which contains $q(U) = \{z : \mathrm{Re}(z) > 0\}\ and\ \phi(w) \neq 0\ when\ w \in q(U)$. Further

$$Q(z) = zq'(z)\phi(z) = \frac{2z}{(1+z)^2}$$

is starlike in *U* and for the function

$$h(z) = \theta(q(z)) + Q(z) = \frac{2z(z+2)}{(1+z)^2} = 2 - \frac{2}{(1+z)^2}$$

we have

$$\mathrm{Re}\left\{\frac{h'(z)}{Q'(z)}\right\} = \mathrm{Re}\left\{\frac{2}{1-z}\right\} > 0,\ z \in U$$

Also *p* is analytic in $U,\ p(0) = q(0) = 1\ and\ p(U) \subset D$ because 0 is not in $p(U)$. Therefore the conditions of lemma 1.1 are satisfied and we obtain the result that if

$$\theta(p(z)) + zp'(z)\,\phi\,(p(z)) = 1 - \left(z\,\frac{g'(z)}{f'(z)}\right)' \prec 2 - \frac{2}{(1+z)^2} = h(z)$$

we have

$$\frac{f'(z)}{g'(z)} = p(z) \prec q(z) = \frac{1+z}{1-z}$$

that is $f \in C^*$

2.2 Example

Let $f(z) = \log_e \dfrac{k(z)}{z},\ g(z) = \dfrac{1+z}{1-z}$ where $k(z) = \dfrac{z}{(1-z)^2}$, the function *f(z)* belongs to the class A and $1 - \left(z\,\dfrac{g'(z)}{f'(z)}\right)' = \dfrac{z(z-4)}{(1-z)^2}$ is subordinated to $2 - \dfrac{2}{(1+z)^2}$. So from Theorem (2.1) $f \in C^*$.

2.3 Theorem

Let $f, g \in A$, If $1 - \left(z\,\dfrac{g'(z)}{f'(z)}\right)' \prec h(z),\ h(0) = 0\ and\ h(z)$ is a convex function in *U* then

$$\frac{g'(z)}{f'(z)} \prec 1 - \frac{1}{z}\int_0^z h(t)dt$$

Proof.

Let $F(z) = \left(z\,\dfrac{g'(z)}{f'(z)}\right)' = 1 - \left(1 - \left(z\,\dfrac{g'(z)}{f'(z)}\right)'\right)\ and\ G(z) = 1 - h(z),\ z \in U$. Then $G(z)$ is a convex univalent in $U,\ G(0) = 1,\ G(z)$ is analytic in $U,\ F(0) = 1$. Further we have that

$$\left(z\,\frac{g'(z)}{f'(z)}\right)' = F(z) \prec G(z) = 1 - h(z)$$

Therefore the condition of lemma 1.2 are satisfied and for n = 0 we obtain

$$\frac{1}{z}\int_0^z F(t)dt \prec \frac{1}{z}\int_0^z G(t)dt$$

If we apply the definitions of $F(z)$ and $G(z)$ in the result above and use the following fact which is true because $F(z)$ is analytic

$$\int_0^z \left(z\,\frac{g'(z)}{f'(z)}\right)' dz = \frac{zg'(z)}{f'(z)}$$

we obtain that

$$\frac{g'(z)}{f'(z)} \prec \frac{1}{z}\int_0^z (1-h(t))\,dt = 1 - \frac{1}{z}\int_0^z h(t)\,dt$$

2.4 Corollary

If $f, g \in A$, and if $\left|1 - \left(z\,\frac{g'(z)}{f'(z)}\right)'\right| < 2,\ z \in U\ then\ f \in C^*$

Proof.

From $\left|1 - \left(z\,\frac{g'(z)}{f'(z)}\right)'\right| < 2,\ z \in U\ because\ h(z) = 2z$ is univalent and

$$1 - \left(0\,\frac{g'(0)}{f'(0)}\right)' = h(0) = 0 \text{ we get that}$$

$$1 - \left(z\,\frac{g'(z)}{f'(z)}\right)' < 2z = h(z)$$

Further, $h(z)$ is convex, so the condition from Theorem 2.3 is satisfied and we obtain

$$\frac{g'(z)}{f'(z)} \prec 1 - \frac{1}{z}\int_0^z h(t)\,dt = 1 - z$$

that is

$$\operatorname{Re}\left\{\frac{g'(z)}{f'(z)}\right\} > 0$$

Because of that, $\operatorname{Re}\left\{\frac{f'(z)}{g'(z)}\right\} > 0$

that is $f \in C^*$.

2.5 Example

The same functions as in the example above, $f(z) = \log_e \frac{k(z)}{z},\ g(z) = \frac{1+z}{1-z}$ can be used to illustrate corollary 2.4

$$\left|1 - \left(z\,\frac{g'(z)}{f'(z)}\right)'\right| = \left|1 - \frac{1}{(1-z)^2}\right| < 2$$

and *f(z)* is close-to-convex

2.6 Corollary

Let $f, g \in A$,

1. If $1 - \left(z\,\frac{g'(z)}{f'(z)}\right)' \prec \frac{\alpha z}{1+z} = h(z),\ 0 \le \alpha \le \frac{1}{2}(1-\ln 2)$ then f is close-to-convex

2. If $\operatorname{Re} - \left\{1 - \left(z\,\frac{g'(z)}{f'(z)}\right)'\right\} < \frac{1}{2}(1-\ln 2) = 1.629445,\ z \in U$ then f is close-to-convex

Proof.

1. From $h(0) = 0$ *and* $h(z)$ is a convex function in the unit disc U and by Theorem 2.3 we get that

$$\frac{g'(z)}{f'(z)} \prec 1 - \frac{1}{z}\int_0^z h(t)\,dt = 1 - 2\alpha + 2\alpha\,\frac{\ln(1+z)}{z} = q(z)$$

Now from

$$\operatorname{Re}\{q(z)\} = 1 - 2\alpha + \frac{2\alpha}{|z|^2}\left[x\ln|1+z| + y\arg(1+z)\right]$$

$$\operatorname{Im}\{q(z)\} = \frac{2\alpha}{|z|^2}\left[x\arg(1+z) - y\ln|1+z|\right]$$

where $z = x + iy$, it follows that $q(U)$ is symmetric with respect to the x-axis and so

$$\operatorname{Re}\{q(z)\} > \min\{q(1), q(-1)\} = q(1) = 1 - 2\alpha + 2\alpha\ln 2 > 0,\ z \in U$$

Thus from $\frac{g'(z)}{f'(z)} < q(z)$ we get that $\operatorname{Re}\left\{\frac{g'(z)}{f'(z)}\right\} > 0,\ z \in U$ *and* $\operatorname{Re}\left\{\frac{f'(z)}{g'(z)}\right\} > 0,\ z \in U$,

that is f is close-to-convex function.

2. $1 - \left(z\,\frac{g'(z)}{f'(z)}\right)'$ is analytic in the unit disc U, $h(z)$ is univalent in U and

$$1 - \left(0\,\frac{g'(0)}{f'(0)}\right)' = h(0) = 0$$

Therefore the condition from (1) of corollary 2.6

$$1 - \left(z\,\frac{g'(z)}{f'(z)}\right)' < \frac{2\alpha z}{1+z} = h(z)$$

is equivalent to

$$1 - \left(z\,\frac{g'(z)}{f'(z)}\right)' \in h(U),\quad z \in U$$

Now from $\operatorname{Re}\{h(e^{i\theta})\} = \alpha$ *and* $h(0) = 0 < \alpha$, we get that $h(z)$ maps the unit disc U into the half plane with real part less than α. So the condition from (1) is equivalent to

$$\operatorname{Re}\left\{1 - \left(z\,\frac{g'(z)}{f'(z)}\right)'\right\} < \alpha,\ z \in U$$

If we put $\alpha = \frac{1}{2}(1 - \ln 2)$ here, using (1) we obtain the statement of (2).

2.7 Example

For $f(z) = \frac{(\log_e z)^2}{2}$ and $g(z) = \log_e z$, then $f(z) = \frac{(g(z))^2}{2}$ and we have that $f, g \in A$, and $1 - \left(z \frac{g'(z)}{f'(z)}\right)' = 1 - z^2 (\log_e z + 1)$. Further, for $z = e^{i\theta}$ we get $\text{Re}\{1 - z^2 (\log_e z + 1)\} = 1 - (\cos^2\theta - \sin^2\theta)$. So from corollary 2.6 (2) we obtain that *f(z)* is close-to-convex function.

In conclusion therefore, we can say that from Theorem 2.1, corollary 2.4, and corollary 2.6, we have used certain properties of $1 - \left(z \frac{g'(z)}{f'(z)}\right)'$ to establish the sufficient condition for *f(z)* to be close-to-convex function.

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