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Application of New Variational Homotopy Perturbation Method For Solving Integro – Differential Equations

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Abstract

This paper discusses the application of the New Variational Homotopy Perturbation Method (NVHPM) for solving integro-differential equations. The advantage of the new Scheme is that it does not require discretization, linearization or any restrictive assumption of any form be fore it is applied. Several test problems are considered and the results obtained are compared in terms of errors obtained with the two convectional variational iteration and Homotopy Perturbation Methods. Also the results obtained compared favourably with the exact solution. In all the examples considered, the results reveal that the proposed method is very efficient, simple and is more user friendly.

Keywords: Variational Iteration Method, Homotopy Perturbation Method, New Variational Homotopy Perturbation Method, Integro-Differential Equations.

1.0 Introduction

There has recently been much attention devoted to the search for reliable and more efficient solution methods for both linear and nonlinear differential equations which appear in diverse physical phenomena in various fields of science and engineering. Hence the method described in this work is been proposed. This research work is designed to be a reliable mixture of two powerful methods. Firstly, He [1-5] developed the variational iteration methods (VIM) for solving linear and nonlinear, initial and boundary value problems. It is worth mentioning that the origin of variational iteration method can be traced back to Inokuti et al [6], but the potential of the technique was explored by He [2-5].

Moreover, He, realized the physical significance of the variational iteration method, its compatibility with physical problems and He applied this promising technique to a wide class of linear and nonlinear, ordinary, partial or stochastic differential equations. Secondly, Homotopy perturbation method (HPM) was proposed by He [7]. This does not require a small parameter in the equation contrary to the traditional perturbation methods. In He’s Homotopy perturbation with an embedding parameter $p \in [0, 1]$, homotopy is constructed. The Homotopy perturbation method gets to the solution with much less computational work. It is based on Taylors series with respect to an embedding parameter. Mathematically speaking, Homotopy perturbation method itself is also a kind of generalized Taylor technique. It can give a very good approximation by means of a few terms, if the initial guess and the auxiliary linear operator are good enough. In later work, Ghorbani [8-9] splits the nonlinear term into series of polynomials calling them He’s polynomials. Recently, Noor and MohyudDim [10] used homotopy perturbation, variational iteration and the iterative methods for solving Boundary Value Problems (BVP).

It is well known that many mathematical formulations of physical phenomena contain integro-differential equations. These equations arise in many fields like fluid dynamics, biological models and chemical kinetics. Integro-differential equations are usually difficult to solve analytically so it is required to obtain an efficient approximate solution. The first order integro-differential equation is given by

$$du / dx = f(x) + \int_0^x \Psi\left(t, u(t), u'(t)\right) dt \qquad (1.00)$$

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were f(x) is the source term and u(x) is the unknown which is to be determined.

The proposed algorithm is expected to provide solution in a rapid convergent series which may lead to a solution in a closed form. In this technique, the correction functional is developed [3 – 6, 11 – 15] and the Lagrange multipliers are calculated optimally via variational theory. The use of Lagrange multipliers reduces the successive application of integral operator and the huge computational work, while still maintaining a very high level of accuracy. Finally, He's polynomials are introduced in the correction functional and the comparison of the like powers of p gives solution of various order.

2.0 Methodology

Basically, we review the conventional variational Iteration and the homotopy perturbation methods for the solution of general differential equations.

(I) Basic Idea of Homotopy Perturbation Method

Linear and Non-linear Phenomena are of fundamental importance in various fields of science and engineering. Most models of real-life problems are still very difficult to solve. Therefore, an approximate analytical solution such as Homotopy Perturbation Method was introduced. To explain this method, we consider the following general non-linear differential equation:

$$A(u) - f(r) = 0,\ r \in \Omega \tag{2.00}$$

with the boundary condition

$$B(u, \delta u/\delta n) = 0,\ r \in \Gamma \tag{2.01}$$

where A, B, f (r) and Γ are a general differential operator, a boundary operator, a known analytical function and the boundary of the domain Ω, respectively.

The operator A can, generally speaking, be divided into parts L and N (say), where L is the linear part, while N is the non-linear part. Equation (2.00) therefore can be rewritten as follows:

$$L(u) + N(u) - f(r) = 0 \tag{2.02}$$

By the Homotopy techniques, we constructed a Homotopy

$\mathrm{v}(r, p) : \Omega x\, [0, 1] \rightarrow R$ which satisfies:

$$H(\mathrm{v}, p) = (1-p)\left[L(\mathrm{v}) - L(u_0)\right] + p\left[A(\mathrm{v}) - f(r)\right] = 0, \tag{2.03}$$

$$p \in [0, 1],\ r \in \Omega$$

or

$$H(\mathrm{v}, p) = L(\mathrm{v}) - L(u_0) + pL(u_0) + p[N(\mathrm{v}) - f(r)] = 0 \tag{2.04}$$

where $p \in [0, 1]$, is an embedding parameter, while u_0 is an initial approximation of equation (2.00) which satisfies the boundary condition. Obviously, from equation (2.03) and (2.04) we will have:

$$H(\mathrm{v}, 0) = L(\mathrm{v}) - L(u_0) = 0 \tag{2.05}$$

$$H(\mathrm{v}, 1) = A(\mathrm{v}) - f(r) = 0, \tag{2.06}$$

The process of changing *p* from zero to unity is just that of changing $H(\mathrm{v}, p)\ from\ L(\mathrm{v}) - L(u_0)\ to\ A(\mathrm{v}) - f(r)$. In topology, this is called deformation, while $L(\mathrm{v}) - L(u_0)$ and $A(\mathrm{v}) - f(r)$ are called homotopic. According to the Homotopy Perturbation method, we can first use the embedding parameter *p* as a small parameter, and assume that the solution of equation (2.03) and (2.04) can be written as a power series in *p*:

$$V = \mathrm{v}_0 + p\mathrm{v}_1 + p^2\mathrm{v}_2 + p^3\mathrm{v}_3 + \ldots \tag{2.07}$$

$$V = \lim_{p \to 1} \mathrm{v}_0 + \mathrm{v}_2 + \mathrm{v}_3 + \ldots \tag{2.08}$$

The combination of the perturbation method and the Homotopy method is called the Homotopy Perturbation Method, which eliminates the drawbacks of the traditional perturbation methods while keeping all its advantages.

The series (2.08) is convergent for most cases. However, the convergent rate depends on the non-linear operator. He [7], made the following suggestions:

The second derivative of N(v) with respect to v must be small because the parameter may be relatively large, i.e. p $\rightarrow$ 1. The norm of $L^{-1}\ \delta N/\delta \mathrm{v}$ must be smaller than one so that the series converges.

The introduction of equation (2.07) into equation (2.03) and (2.04) and comparison of like powers of *p* gives solution of various orders.

(II) Basic Idea of Variational Iteration Method

To illustrate the basic ideas of variational iteration method, we consider the following differential equation:

$LU + NU = g(t)$ (2.09)

where

L is a linear operator

N is a nonlinear operator

g(t) is the homogenous term

According to variational iteration method, we can construct a correction functional as follows:

$$u_{n+1}(t) = u_n(t) + \int_0^1 \lambda \left[LU_n(\tau) + N\overline{U}_n(\tau) - g(\tau)\right] d\tau \qquad (2.10)$$

where λ is a general Lagrangian multiplier which can be identified optimally via the variational theory. The subscript 'n' indicates the nth approximation and $\overline{U}_n$ is considered as a restricted variation i.e. $\delta\overline{U}_n = 0$. The solution of the linear problems can be solved in a single iteration step due to the exact identification of the Lagrange multiplier.

The successive approximation $u_{n+1}, n \geq 0$ of the solution U will be readily obtained upon using the determined Lagrange multiplier and any selective function U_0. Consequently, the solution is given by $U = \lim_n \to \infty\, U_n$.

2.1 Implementation of NVHPM to Integro-Differential Equations

In this section, we extend the new scheme to integro-differential equations. The first order integro-differential equation is given by

$$du/dx = f(x) + \int_0^x \Psi(t, u(t), u'(t))\, dt \qquad (2.11)$$

where *f(x)* is the source term and *u(x)* is the unknown which is to be determined via the new variational Homotopy perturbation method (NVHPM).

The correction functional according to variational iteration method can be constructed as follows:

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(s)\left[(u_n)_s - f(s) - \int_0^s \Psi\, (t, \overline{u}(t), \overline{u}'(t))dt\right] ds \qquad (2.12)$$

where $\overline{u}_n$ is considered as restricted variations, which means $\overline{u}_n = 0$. To find the optimal $\lambda(s)$, we proceed as follows:

$$\delta u_{n+1}(x) = \delta u_n(x) + \delta \int_0^x \lambda(s)\left[(u_n)_s - f(s) - \int_0^s \Psi\, (t, \overline{u}(t), \overline{u}'(t))dt\right] ds \qquad (2.13)$$

And consequently,

$$\delta u_{n+1}(x) = \delta u_n(x) + \delta \int_0^x \lambda(s)(u_n)_s\, ds \qquad (2.14)$$

which results in,

$$\delta u_{n+1}(x) = \delta u_n(x) + \lambda(s)\, \delta u_n(x) - \int_0^x \delta u_n(x)\lambda'(s)\, ds \qquad (2.15)$$

The stationary conditions can be obtained as follows

$$\lambda'(s) = 0 \text{ and } 1 + \lambda(s)\big|_{s=t} = 0 \qquad (2.16)$$

The lagrange multipliers, therefore, can be identified as

$$\lambda(s) = -1$$

And the iteration formula is given as

$$u_{n+1}(x) = u_n(x) - \int_0^x \left[(u_n)_s - f(s) - \int_0^s \Psi\, (t, \overline{u}(t), \overline{u}'(t))dt\right] ds \qquad (2.17)$$

Now, we implement the New Variational Homotopy Perturbation Method on eqn. (2.17) to obtain

$$\sum_{n=0}^{\infty} p^n U_n = U_0(x) - p\int_0^x \left[\sum_{n=0}^{\infty} p^n U_n - f(s) - \int_0^s \left(\sum_{n=0}^{\infty} p^n \Psi_n\right) dt\right] ds \qquad (2.18)$$

The expansion of eqn. (2.18), gives:

$$u_0 + pu_1 + p^2u_2 + \ldots = u_0(x) - p\int_0^x \Big[\left(u_0 + pu_1 + p^2u_2 + .\right)_s - f(s)$$
$$-\int_0^s \left(\Psi_0 + p\Psi_1 + \ldots\right) dt\Big] ds \tag{2.19}$$

The comparison of the coefficients of like powers of *p* gives solutions of various orders of the form:

$$p^0 : \; u_0 = u_0(x)$$
$$p^1 : \; u_1 = -\int_0^x \left[\frac{\delta u_0}{\delta s} - f(s) - \int_0^s \Psi_0 \, dt\right] ds$$
$$p^2 : \; u_2 = -\int_0^x \left[\frac{\delta u_1}{\delta s} - \int_0^s \Psi_1 \, dt\right] ds \tag{2.10}$$
$$p^n : \; u_n = -\int_0^x \left[\frac{\delta u_{n-1}}{\delta s} - \int_0^s \Psi_{n-1} \, dt\right] ds$$

3.0 Numerical Examples

Example 3.1:

We first consider the nonlinear integro-differential equation given by Batiha et al [16]:

$$u'(x) = -1 + \int_0^x u^2(t) dt \tag{3.01}$$

for $x \in [0, 1]$ with the boundary condition $u(0) = 0$.

We construct a correction functional according to the variational iteration method as follows:

$$u_{n+1}(x) = u_n(x) - \int_0^x \left[(u_n)_s(s) + 1 - \int_0^s u^2(t) dt\right] ds \tag{3.02}$$

Now, we implement the New Variational Homotopy Perturbation Method on eqn. (3.02), to obtain

$$u_0 + pu_1 + p^2u_2 + \ldots = u_0(x) - p\int_0^x \left[(u_0 + pu_1 + \ldots)_s + 1 - \int_0^s (u_0 + pu_1 + \ldots)^2 dt\right] ds \tag{3.03}$$

The comparison of the coefficient of like powers of *p* gives solution of various orders as follows:

$$p^0 : \; u_0 = u_0(x) = -x$$
$$p^1 : \; u_1 = -\int_0^x \left(\frac{\delta u_0}{\delta s} + 1 - \int_0^s u_0^2(t) dt\right) ds \tag{3.04}$$
$$= -x + \frac{x^4}{2}$$

Example 3.2:

Now we find the approximate solution of the integro-differential equation given by Batiha et al [16]:

$$u'(x) = +1 + \int_0^x u(t)\frac{du(t)}{dt} dt \tag{3.05}$$

for $x \in [0, 1]$ with the boundary condition $u(0) = 0$.

We construct a correction functional according to the variational iteration method as follows:

$$u_{n+1}(x) = u_n(x) - \int_0^x \left[(u_n)_s(s) - 1 - \int_0^s u(t)\frac{du(t)}{dt} dt\right] ds \tag{3.06}$$

Now, we implement the new variational Homotopy perturbation method on eqn. (3.06), to obtain

$$u_0 + pu_1 + p^2u_2 + \ldots = u_0(x) - p\int_0^x \Big[(u_0 + pu_1 + \ldots)_s - 1$$
$$-\int_0^s (u_0 + pu_1 + \ldots)(u_0 + pu_1 + \ldots)_t \, dt\Big] ds \tag{3.07}$$

The comparison of the coefficient of like powers of *p* gives solution of various orders as follows:

$$p^0 : \; u_0 = u_0(x) = x$$

$$p^1: \; u_1 = -\int_0^x \left[-1 - \int_0^s u_0 u_0' dt \right] ds$$

$$= x + \frac{x^3}{2} \qquad (3.08)a$$

$$p^2: \; u_2 = -\int_0^x \left[-1 - \int_0^s \left(u_0 \, u_1' + u_0' \; u_1 \right) dt \right] ds$$

(3.08)b

$$= \quad x \quad + \quad x^3 \; + \; x^5$$

Example 3.3:

We consider the second order non-linear integro-differential equation given by Ivaz et al [12]:

$$u''(x) + xu'(x) - xu(x) - \int_0^1 \sin x \ell^{-2t} u^2(t) dt = e^x - \sin x \qquad (3.09)$$

for $x \in [0, 1]$ with the boundary condition $u(0) = 1, u'(0) = 1$ with the exact solution $u(x) = e^x$.

We construct a correction functional according to the variational iteration method as follows:

$$u_{n+1}(x) = u_n(x) + \int_0^x (s-t) \left[\frac{\delta^2 u_n}{\delta s^2} + x \frac{\delta u_n}{\delta s} - u(x) - e^x + \sin x - \int_0^1 \sin x e^{-2t} u^2(t) dt \right] ds$$

hence

$$u_{n+1}(x) = u_n(x) - \frac{t^2}{2} \int_0^x \left[\frac{\delta^2 u_n}{\delta s^2} + x \frac{\delta u_n}{\delta s} - u(x) - e^x + \sin x - \int_0^1 \sin x e^{-2t} u^2(t) dt \right] ds \qquad (3.10)$$

Now, we implement the new variational Homopoty perturbation method on eqn. (3.10), to obtain:

$$u_0 + pu_1 + p^2 u_2 + \ldots = u_0(x) - \frac{pt^2}{2} \int_0^x \left[\frac{\delta^2 u_n}{\delta s^2} + x \frac{\delta u_n}{\delta s} - u(x) - e^x \right.$$

$$\left. + \sin x - \int_0^1 \sin x e^{-2t} u^2(t) \, dt \right] ds \qquad (3.11)$$

The comparison of the coefficients of like powers of *p* gives solution of various orders as follows:

$$p^0: \; u_0 = u_0(x)$$

$$p^1: \; u_1 = -\frac{t^2}{2} \int_0^x \left[\frac{\delta^2 u_0}{\delta s^2} + x \frac{\delta u_0}{\delta s} - u_0(x) - e^x + \sin x \right.$$

$$\left. - \int_0^1 \sin x e^{-2t} u_0^2(t) \, dt \right] ds \qquad (3.12)$$

$$p^2: \; u_2 = -\frac{t^2}{2} \int_0^x \left[\frac{\delta^2 u_1}{\delta s^2} + x \frac{\delta u_1}{\delta s} - u_1(x) - e^x + \sin x \right.$$

$$\left. - \int_0^1 \sin x e^{-2t} u_1^2(t) \, dt \right] ds$$

which implies,

$$u_0 \quad = \quad 1 + x$$

$$u_1 = \frac{t^2}{2} \left[x^2 \left(1 - \sin x . e^{-2} + \sin x \right) + x^3 \left(1 - \frac{1}{2} \sin x . e^{-2} + \frac{1}{2} \sin x \right) \right.$$

$$\left. + x \left(e^x - \frac{1}{2} \sin x - \frac{1}{2} \sin x . e^{-2} \right) \right]$$

Since the solution can be ascertaineds after the first iteration, hence

$$U = 1 + x + \frac{t^2}{2}\left[x\left(e^x - \frac{1}{2}\sin x - \frac{1}{2}\sin x.e^{-2} \right) + x^2 \left(1 - \sin x.e^{-2} + \sin x\right) + x^3 \left(1 - \frac{1}{2}\sin x.e^{-2} + \frac{1}{2}\sin x \right)\right] \quad (3.13)$$

Example 3.4:

We again consider the second order nonlinear integro-differential equation given by Ivaz et al [12]:

$$u''(x) + u(x) - \int_0^{\frac{\pi}{2}} x\cos t u^2(t)\, dt = -\frac{x}{3} \quad (3.14)$$

$$u(0) = 0,\ u'(0) = 1$$

with the exact solution $u(x) = \sin x$.

We construct a correction functional according to the variational iteration method as follows:

$$u_{n+1}(x) = u_n(x) + \int_0^x (s-t)\left[\frac{\delta^2 u_n}{\delta s^2} + u_n - \int_0^{\frac{\pi}{2}} x\cos t u_n^2(t)dt + \frac{x}{3} \right] ds$$

and

$$u_{n+1}(x) = u_n(x) - \frac{t^2}{2}\int_0^x \left[\frac{\delta^2 u_n}{\delta s^2} + u_n(x) - \int_0^{\frac{\pi}{2}} x\cos t u_n^2(t)dt + \frac{x}{3} \right] ds \quad (3.15)$$

Now, we implement the new variational Homotopy perturbation method on eqn. (3.15), to obtain:

$$u_0 + pu_1 + \ldots = u_n(x) - \frac{pt^2}{2}\int_0^x \left[\frac{\delta^2 u_n}{\delta s^2} + u_n(x) + \frac{x}{3} - \int_0^{\frac{\pi}{2}} x\cos t u_n^2(t)dt \right] ds \quad (3.16)$$

The comparison of the coefficient of like powers of *p* gives solution of various orders as follows:

$$p^0 : \ u_0 = u_0(x)$$

$$p^1 : u_1(x) = -\frac{t^2}{2}\int_0^x \left[\frac{\delta^2 u_0}{\delta s^2} + u_0(x) + \frac{x}{3} - \int_0^{\frac{\pi}{2}} x\cos t u_0^2(t)dt \right] ds \quad (3.17)$$

$$p^2 : u_2(x) = -\frac{t^2}{2}\int_0^x \left[\frac{\delta^2 u_1}{\delta s^2} + u_1(x) + \frac{x}{3} - \int_1^{\frac{\pi}{2}} x\cos t u_1^2(t)dt \right] ds$$

Which implies,

$$u_0 = x$$

$$u_1 = -\frac{t^2}{2}\int_0^x \left(x^2 + \frac{x^2}{3} - x^4 \sin x \right) \quad (3.18)$$

Since the solution can be ascertained after the first iteration, hence

$$U = x - \frac{t^2}{2}\left(x^2 + \frac{x^2}{3} - x^4 \sin x \right) \quad (3.19)$$

4.0 Results

In the section we present our results in tabular form for easy comparison.

Table 4.1: Comparison between HPM/VIM and NVHPM results for Example 3.1

X	VIM/HPM	NVHPM	ERROR
0.000	0.000	0.000	0.000
0.094	-0.094	-0.094	3.220×10^{-5}
0.219	-0.219	-0.218	9.550×10^{-4}
0.313	-0.312	-0.308	3.975×10^{-3}
0.406	-0.404	-0.393	1.114×10^{-2}
0.500	-0.495	-0.469	2.607×10^{-2}

Table 4.2: Comparison between HPM/VIM and NVHPM results for Example 3.2

X	VIM/HPM	NVHPM	ERROR
0.000	0.000	0.000	0.000
0.094	0.094	0.094	4.061×10^{-4}
0.219	0.219	0.228	5.046×10^{-3}
0.313	0.312	0.328	1.446×10^{-2}
0.406	0.408	0.440	3.122×10^{-2}
0.500	0.505	0.563	5.720×10^{-2}

Table 4.3: Comparison between EXACT and NVHPM results for Example 3.3

X	EXACT	NVHPM	ERROR
0.100	1.105	1.100	4.586×10^{-3}
0.200	1.221	1.217	4.705×10^{-3}
0.300	1.350	1.328	2.199×10^{-2}
0.400	1.492	1.441	5.044×10^{-2}
0.500	1.649	1.558	9.110×10^{-2}
0.600	1.822	1.677	1.451×10^{-1}
0.700	2.014	1.800	2.136×10^{-1}
0.800	2.226	1.927	2.982×10^{-1}
0.900	2.460	2.0560	4.001×10^{-1}

Table 4.4: Comparison between EXACT and NVHPM results for Example 3.4

X	EXACT	NVHPM	ERROR
0.100	0.100	0.100	1.000×10^{-4}
0.200	0.199	0.200	1.060×10^{-3}
0.300	0.296	0.300	3.880×10^{-3}
0.400	0.389	0.399	9.510×10^{-3}
0.500	0.480	0.498	1.890×10^{-2}
0.600	0.565	0.598	3.296×10^{-2}
0.700	0.644	0.697	5.251×10^{-2}
0.800	0.717	0.796	7.837×10^{-2}
0.900	0.783	0.895	1.113×10^{-1}

5.0 Conclusion

In this paper, a New Variational Homotopy Perturbation Method has been successfully applied to find the solution of Integro.-differential equations and the results obtained compared favourably with the two convectional variational iteration and Homotopy Perturbation Method and the exact solution. It can be concluded that the NVHPM is a very powerful and efficient technique for finding approximate solutions for wide classes of problems. It is worth mentioning that the Method is computational cost friendly.

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