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On The Relative Controllability Perturbation of Nonlinear Functional Differential Systems with Implicit Derivative.

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Abstract

In this paper, we provide sufficient conditions for the relative controllability of perturbation of nonlinear function differential systems with implicit derivative. This result is obtained using Darbo's fixed point theorem.

1.0 Introduction

Nonlinear systems present a challenge but fascinating area of study in mathematical control theory. They represent better approximations of real life dynamics and pose the obvious difficulty of not lending themselves readily to systematic and precise procedures of tackling controllability problems. However, several studies have been conducted on perturbation of linear systems. In most of these studies results were obtained by placing growth conditions and continuity conditions on the perturbation functions; and the Schauder's fixed point approach was greatly used. Eke [1], Onwuatu [2], Balachandran and Dauer [3] have shown that a bounded linear perturbation is controllable provided, the linear base is controllable. Nonlinear ordinary systems have been studied by Klamka [4], and Onwuatu [5]. Balachandran and Dauer [3], Dacka [6] and Balachandran [7] have considered perturbations of nonlinear ordinary systems with implicit derivative. In these studies, together with the dynamics modelled by Chukwu [8,9], the systems are multi-parameter dependent, necessitating the redefinition of the fundamental matrix solution and the controllability grammian to take care of these systems' varying arguments. In [4] and [10] nonlinear systems with infinite delays are considered in various spaces. The use of Darbo's fixed point theorem in[6,7] imposed the calculation of the common modulus of continuity of functions in a set and consequently the measure of non-compactness of the set to take care of the rigors introduced by the presence of implicit derivative. From the foregoing, the relative controllability of nonlinear functional differential systems with distributed delays and with nonlinear base remains unsettled especially for some systems with implicit derivative, the investigation of which is the main objective of our research here

2.0 Notations And Preliminaries

Let $E = (-\infty, \infty)$ and E^n be the n dimensional Euclidean space with norm $|\cdot|$ The symbol $C = C([-h,0], E^n)$ denotes the space of continuous functions mapping the interval $[-h,0]$, $h > 0,\ h \in E$ into E^n with the supremum norm $\| \bullet \|$ defined by $\| \phi \| = \sup |\phi(\theta)|$; $\phi \in C, -h \le \theta \le 0$

While $C' = C'([-h, 0], E^n)$ denotes the space of differentiable functions mapping the interval $[-h, 0]$ into E^n with sup norm $\| \phi \| = \sup (|\phi(\theta)| + |\phi'(\theta)|)$; $\phi \in C$.

Let $(X, \| \bullet \|)$ be a Banach space and Q a bounded subset of X. The measure of non-compactness of Q, given as μ (Q) = inf {r > o: Q can be covered by a finite number of balls of radii less than r}(see [1])

For the space of continuous functions $C([t_o, t_1], E^n)$, the measure of non-compactness of a set Q is given by

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$$\mu(Q) = \tfrac{1}{2} W_0(Q) = \tfrac{1}{2} \lim_{h \to 0^+} W(Q,h)$$

where W(Q,h) is the common modulus of continuity of the functions which belong to set Q, that is

$$W(Q,h) = \sup_{x \in Q} \{\sup |x(t) - y(s)| : |t - s| \le h\}$$

For the space of differentiable functions $C'([-h, 0], E^n)$, we have

$$\mu(Q) = \tfrac{1}{2} W_o (DQ)$$

where

$$DQ = \{x' : x \in Q\}.$$

If $t \in [t_0, t_1]$, we let $x_t \in C$ be defined by $X_t(s) = X(t+s), s \in [-h, 0]$

Also, for functions $u : [t_o - h, t_1] \to E^m, h > 0, and\ t \in [t_o, t_1]$ then u_t denotes the functions on [-h, 0] defined by $u_t(s) = u(t+s)\ for\ s \in [-h, 0]$.

Assume the integral is in the Lebesgue Stieltjes sense, and consider the system of interest

$$\dot{X}(t) = L(t, x_t, u_t) x_t + B(t, x_t, u_t) u_t + f\left(t, x_t \dot{x}_t, u_t\right) \tag{2.1}$$

with the following basic assumptions:

$$L(t, \phi, \psi)\, x_t = \int_{-h}^{0} d\eta(s, \phi, \psi)\, x(t+s)$$

where the nxn matrix function $\eta(t, s, \phi, \psi)$ is measurable in $(t, s) \in$ ExE, and normalized so that

$$\eta(t, s, \phi, \psi) = 0; \quad s \ge 0\ for\ all\ \phi, \psi$$

$$\eta(t, s, \phi, \psi) = \eta(t, -h, \phi, \psi)\ for\ all\ s \le -h$$

$\eta(t, s, \phi, \psi)$ is continuous from the left in s on (-h, 0) and has bounded variation in s on [-h, 0] for each t,ψ,ϕ and there is an integrable function m(t) such that

$$|L(t, s, \phi, \psi)\, x_t| \le m(t)\, \|x_t\|,\ for\ all\ t \in (-\infty, \infty), and\ \phi, x_t \in C$$

We assume $L(t, s, \phi, \psi)$ is continuous. The nxn matrix $B(t, x_t, u_t)$ given by

$$B(t, x_t, u_t) u_t \quad = \quad \int_{-h}^{0} ds\ H\,(t, x(t+s), u(t+s))\, u(t+s)$$

is continuous on the variables and of bounded variation in s on [-h, 0] also. The function f is continuous and satisfies the Lipschitz condition in all its arguments. Enough smoothness conditions on L and f are imposed to ensure the existence of solution of systems (2.1)and the continuous dependence of same on initial data.

Definition2.1

The set $y(t)=\{x(t), x_t, u_t\}$ is said to be the complete state of system (2.1) [10]

Definition 2.2[10]

System (2.1) is said to be relatively controllable on $[t_0, t_1]$, if for every initial complete state $y(t_0)$ and every $x_1 \in E^n$, there exists a control u(t) defined on $[t_0, t_1]$ such that the corresponding trajectory of system (2.1) satisfies $x(t_1) = x_1$.

Definition2.3 (Darbo's fixed point theorem) [1]

If s is a non-empty, bounded, closed, convex subset of x and P: S → S is a continuous mapping such that for any Q ∈ S, we have

$$\mu(pQ) \le k\mu(Q)$$

where k is a constant $0 < k \le 1$, then P has a fixed point.

3.0 Main Results

To solve the relative controllability problem for the system (2.1) we consider the linear approximation of

$$\dot{X} = L(t, x_t, u_t)x_t + B(t_1, x_t, u_t)u_t \tag{3.1}$$

by specifying some arguments of L and B to have

$$\dot{X}(t) = L(t, z, \mathrm{v})\, x_t + B(t, z, \mathrm{v})\, u_t \tag{3.2}$$

where the arguments x_t, u_t of L, and B have been replaced by specified functions $z \in C'$, $\mathrm{v} \in C$. System (2.1) can thus be approximated by

$$\dot{X}(t) = L(t, z, \mathrm{v})\, x_t + B(t, z, \mathrm{v})\, u_t + f(t, x_t, \dot{x}\, u_t) \tag{3.3}$$

for each (z, v) ∈ $C' \ X \ C$. One can deduce the variation of parameter for system (3.3) using the unsymmetric Fubini theorem.

Let X(t,s) = X(t, s, z, v)

be the transition matrix for the system

$$\dot{x}(t) = L(t, z, \mathrm{v})x_t \tag{3.4}$$

so that

$$\frac{\partial}{\partial t} X(t,s) = L(t, z, \mathrm{v})\, X(t,s)$$

where

$$X(t,s) = \begin{cases} 0 & s - h \le t < s \\ 1 & t = s \end{cases}$$

and where

$$X_t(\bullet, s)(\theta) = X(t+\theta, s), -h \le \theta \le 0$$

The solution of system (3.3) is given by

$$X(t) = x(t, t_0, \theta, 0) + \int_{t_0}^{t} X(t,s) \ \left[\int_{-h}^{0} d\theta\ H(s, z(\theta))\, \mathrm{v}(\theta) u\, (s+\theta) \right] ds$$

$$+ \int_{t_0}^{t} X(t,s)\, f\left(s, X_s, \dot{X}(s), u(s) \right) ds \tag{3.5}$$

Using the unsymmetrical Fubini theorem as in Klamka [9] which gives impetus to the change of the order of integration, (3.5) can be written as

$$X(t_1) = x(t_1, t_0, \theta, 0) + X(t_1, t_0) \int_{t0}^{t} \int_{-h}^{0} X(t_0, s-\theta) \quad d\theta [H(s-\theta, z(\theta), \mathrm{v}(\theta)] u_{t0}) ds$$

$$+ \quad X(t_1, t_0) \int_{t_0}^{t_1} \left(\int_{-h}^{0} X(t_0, s-\theta) d\theta \left[\hat{H}(s-\theta, z(\theta), \mathrm{v}(\theta)) \right] \right) u(s) ds \tag{3.6}$$

$$+ \ X(t_1, t_0) \int_{t_0}^{t_1} X(t_0, s)\ f(s, X_s, X(s), u(s)) ds$$

where

$$\hat{H}(s, z, \mathrm{v}) = \begin{cases} H(s, z, \mathrm{v}) & \text{for } s \le t \\ O & \text{for } s > t. \end{cases}$$

Let us now define the following notation at $t = t_1$

$$g(t_1) = g(Y(t_0),\ X(t_1), z, \text{v}) = X_1 - X(t_1, t_0, \theta, 0)$$

$$- \int_{t_0}^{t_1} \left\{ \int_{-h}^{0} X(t_1, s-\theta) d\theta [H(s-\theta, z(\theta), \text{v}(\theta)] u_{t_0} \right\} ds - \int_{t_0}^{t_1} X(t_1, s)\ f(s, X_s, X(s), u(s)) ds \quad (3.7)$$

From (3.6), set z to be

$$Z(t_0,\ s,\ z,\ \text{v}) = \int_{t_0}^{t_1} X(t_0, s-\theta),\ z(\theta),\ v(\theta)) \quad (3.8)$$

Thus, the controllability grammian of system (3.2) at time t_1 is

$$W(t_0, t_1) = w(t_0,\ t_1,\ z,\ \text{v}) = \int_{t_0}^{t_1} Z(t_0, s, z, \text{v})\ \ Z^T(t_0, s, z, \text{v})\, ds \quad (3.9)$$

where T denotes matrix transpose.

Relative Controllability Results

Given the system (3.3)

$$\dot{X}(t) = L(t, z, \text{v}) x_t + B(t, z, \text{v}) u_t + f(t, x_t, \dot{x}_t\ u_t)$$

with conditions as spelt out above and where L,B,f are continuous functions in all their variables; and that

$$\left| L\ \ (t, z, \text{v})\ X_t \right| \le m(t) \|\ X_t\ \| \quad (3.10)$$

where m(t) is an integrable function and B(t,z,v) is of bounded variation in s on [-h, 0]. The function f satisfies the Lipschitz condition with respect to the state variable, and the response is uniquely determined by any control. Furthermore,

(a) $\|\ L(t, z, \text{v})\ \| \le$ M for each $s \in$ [-h, 0]

(b) $\|\ L(t, z, \text{v})\ \| \le$ N for each $s \in$ [-h, 0]

(c) $\|\ f(t, x_t, \dot{x}_t, u_t)\ \| \le$ K for each $t \in [t_0, t_1]$

with $z \in C', u \in C\ ([t_0, t_1], E^m)$, where M, N and K are some positive constants. Also for every $x, \dot{x} \in C', u \in C \text{ and } t \in [t_0, t_1]$

(d) $$\left|\ f\left(t, x_t, \dot{x}_t, u_t\right) - f\left(t, y_t, \dot{y}_t, \text{v}_t\right) \right| \le k\ |\ x_t - y_t\ | \quad (3.11)$$

where k is a positive constant such that $0 < k \le 1$

Theorem 3.1

Assume that

Inf det $W(t_0, t_1, z, \text{v}) > 0$ (3.12)

for $z \in C'$,

then system (3.3) is relatively controllable on $[t_0, t_1]$,

Proof

Define the control *u(t)* for $t \in [t_0, t_1]$ as follows

$$u(t) = Z^T(t_0, s, z, \text{v}) W^{-1}(t_0, t_1, z, \text{v}) g\ (Y(t_0),\ X(t_1),\ z,\ \text{v}) \quad (3.13)$$

where $Y(t_0)$ and $x(t_1) = x_1 \in E^n$ are chosen arbitrarily. The inverse of $W(t_0, t_1)$ is possible by condition (3.12). Substituting (3.13) into (3.6) to replace *u(t)* and using (3.7) and (3.9) it is clear that the control *u(t)* defined by (3.13) steers the initial complete state $Y(t_0)$ to the final state $x(t_1) = x_1 \in E^n$. The actual substituting of (3.13) into (3.6) yields

$$X(t) = x(t_1, t_0, \theta, 0) + \int_{t_0+s}^{t_0} \left\{ \int_{-h}^{0}\ X(t, s-\theta) d\theta [H(s-\theta, z(\theta), \text{v}(\theta)] u_{t_0} \right\} ds\ +$$

$$+ \int_{t_0}^{t_1} \left(\int_{-h}^{0}\ X(t, s-\theta) d\theta \left[\hat{H}(s-\theta, z(\theta), \text{v}(\theta)\right] \left[\ X^T(t_0, s, z, \text{v}) W^{-1}(t_0, t_1) g(t_1)\right]\right.$$

$$+ \int_{t_0}^{t} X(t,s)\, f\left(s, X_s, \dot{X}_s, u_s)\right)ds \qquad (3.14)$$

Consider the right hand side of (3.14) as a nonlinear operator which maps the Banach space $C'\left([-h, 0], E^n\right)$ into itself. Hence we can write (3.14) as

$$X(t) = T(x)(t) \qquad (3.15)$$

This operator is continuous.
Define the nonempty closed, Convex subset G by

$$G = \left\{ x : x \in C'\left([-h, 0], E^n, \| x \| \leq N_1, \| Dx \| \leq N_2, \text{ where } Dx = \dot{x} \right\} \qquad (3.16)$$

and the positive real constants N_1 and N_2 are given by

$$N_1 = |\Phi(t_0)| \exp M(t_1 - t_0) + a + (t_1 - t_0) b^2 c k_1 + k(t_1 - t_0) \exp M(t_1 - t_0)$$

$$N_2 = MN_1 + bcNk_1k_2 + k$$

$$K_1 = |X_1| + |\Phi(t_0)| \exp M(t_1 - t_0) + a + 2k(t_1 - t_0) \exp M(t_1 - t_0)$$

$$K_2 = \max \text{ variation } H(t, s, z, \mathrm{v})$$

$$\text{for } t, \quad s \in [-h, 0]$$

a = supremum $\left\| \int_{t_0}^{t_1} \left\{ \int_{-h}^{0} X(t, s-\theta) d\theta [H(s-\theta, z(\theta), v(\theta)] u_{t_0} \right\} ds \right\| ; t \in [t_0, t_1]$

$$b = \sup_{z \in C'} \| Z(t, s, z, \mathrm{v}) \|$$

$$c = \sup_{z \in C'} \| W^{-1}(t_0, t_1, z, \mathrm{v}) \|$$

The constants a, b, c and k_2 exist since the Lebesgue Stieltjes integral with respect to the variable θ is finite.
The operator T maps G onto itself. As clearly seen, all the functions T(x(t)) with x ∈ G are equicontinuous since they all have uniformly bounded derivatives. Now, we shall find an estmate of the modulus of continuity of the function.

DT (x)(t) for t, s ∈ [t_0, t_1]

$$| DT(x)(t) - DT(y)(s) | \leq \left(m(t) \| X_t \| - m(s) \| y_s \| \right) +$$

$$+ \int_{-h}^{0} d\theta H(t, z(t+\theta), \mathrm{v}(t+\theta))\, u(t+\theta) - d\theta\, H(s, z(s+\theta), \mathrm{v}(s+\theta)) u(s+\theta) +$$

$$f\left(t, x_t, \dot{x}_t, u_t\right) - f\left(s, y_s, \dot{y}_s, u_s\right) \qquad (3.17)$$

The first two terms of the right hand side of inequality (3.17) can be estimated as $\beta_0\left(| t-s |\right)$ where β_0 is a non – negative function such that

$$\lim_{h \to 0} \beta_0(h) = 0$$

In the same manner, we find that the other term on the right of (3.17) can be estimated from condition (3.11) as

$$k\, | x(t) - y(s) | + \beta_1\left(| t-s |\right)$$

Letting $\beta = \beta_0 + \beta_1$ we finally obtain

$$| DT(x)(t) - DT(y)(s) | \leq k\, | x(t) - y(s) | + \beta\left(| t-s |\right)$$

Hence, we conclude for any set Q∈ G

$$\mu(TQ) \leq k\mu(Q).$$

Having met all the requirements for the application of Darbo's fixed point theorem, we invoke it. Consequently, by Darbo's fixed point theorem, the operator T has at least one fixed point, therefore, there exists a function $x \in C'([-h, 0], E^n)$ such that

$$X(t) = T(x)(t) \tag{3.18}$$

Differentiating with respect to t, we see that x(t) given by (3.18) is a solution to system (3.3) for the control u(t) given by (3.13). The control u(t) steers the system (3.3) from the initial complete state $Y(t_0)$ to x_1, then by definition 2.2, the system (3.3) is relatively controllable on $[t_0, t_1]$.

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