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An EOQ Model for Delayed Deteriorating Items with Linear Time Dependent Holding Cost

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Abstract

An EOQ model for delayed deteriorating items with linear time dependent holding cost is considered in this paper. This is a little deviation from most inventory models that consider the holding cost to be constant. In this paper, permissible delay in payment is not considered rather the payment is made immediately the replenishment of the inventory is made. The optimal cycle length that gives the minimum total inventory cost was at the end determined and seven numerical examples are given.

1.0 Introduction

The optimal replenishment policy for an Inventory Economic Order Quantity (EOQ) model depends on several costs such as the ordering cost, inventory holding cost, cost of deteriorated items and so on. A lot of work was carried out over the years on the inventory of decaying or deteriorating items. The decaying inventory model was first considered by Ghare and Shrader [1] who developed a model for exponential decaying inventory. Many inventory decaying models were developed based on this maiden research. Much work has also been carried out to extend the EOQ models in order to accommodate time-varying demand, linear trended demand, stock dependent demand and so on.

The first work to solve analytically the EOQ model with linear increasing demand was carried out by Donaldson [2]. Murdeshwar [3] developed an inventory replenishment policy model for linearly increasing demand with shortages. Goyal [4] developed a model on the heuristic for replenishment of trended inventories considering shortages. Goswami and Chaudhuri [5] constructed an EOQ model for inventory items with a linear trend in demand and finite replenishment considering shortages. Datta and Pal [6] reconsidered the model developed by Goswami and Chaudhuri [5] with the restriction that replenishment intervals follow the arithmetic progression pattern. Goh [7] developed a model on the generalized EOQ model for deteriorating items where the demand rate, deteriorating rate, holding cost and ordering cost are all assumed to be continuous functions of time. Giri et.al [8] constructed an EOQ model for deteriorating items with time varying demand and costs. Musa and Sani [9] developed an EOQ model of delayed deteriorating items with no constraint in retailers' capital. This is a situation where items that are stocked do not start deteriorating until after some time. Musa and Sani [10] extended the model developed in [9] to allow for shortages. Musa and Sani [11] constructed a model of delayed deteriorating items under permissible delay in payments. All the above listed models except Goh [7] and Giri et.al [8] considered the inventory holding cost per item per unit time to be a known constant whereas in the case of Goh [7] and Giri et.al [8], the holding cost is considered to be a linear function of time.

In this paper an attempt is made to develop an inventory model for delayed deteriorating items with a linear time dependent holding cost. The Paper is an extension of Musa and Sani [9]. The work considers the situation where the retailer does not allow for permissible delay in settling the replenishment account as in the case of some inventory deteriorating models. The customer is expected to pay for the items as soon as they are received in the inventory. The use of linear time dependent holding cost is justified by the fact that price indices are not in all cases constant but they rather increase with the passage of time.

2.0 Assumptions and Notations

In formulating the mathematical model, the following notations and assumptions are employed:

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2.1 Assumptions

(i) Inventory replenishment is instantaneous (ii) Unconstrained suppliers capital
(iii) There are no shortages (iv) Lead time is negligible

NOTATIONS

β_1 The demand rate during the period before the deterioration sets in
β_2 The demand rate after the deterioration sets in
I_0 The initial level of inventory
T The inventory cycle length
T_1 The time the deterioration sets in
T_2 The difference between the cycle length T and the time the deterioration sets in
C The unit cost of the item
A The ordering cost per order
i The inventory carrying charge per unit per unit time
θ The rate of deterioration
I_d The inventory level at the time the deterioration sets in
$h(t)$ The inventory holding cost, $h(t)=\alpha_1+\alpha_2 t$
I_0 C_H The total inventory holding cost in a cycle
$I(t)$ The level of inventory at any time t before deterioration sets in
$I_d(t)$ The level of inventory at any time t after deterioration sets in
$N(d_t)$ The number of items that deteriorate in the interval $[T_1,T]$

3.0 The Mathematical Model

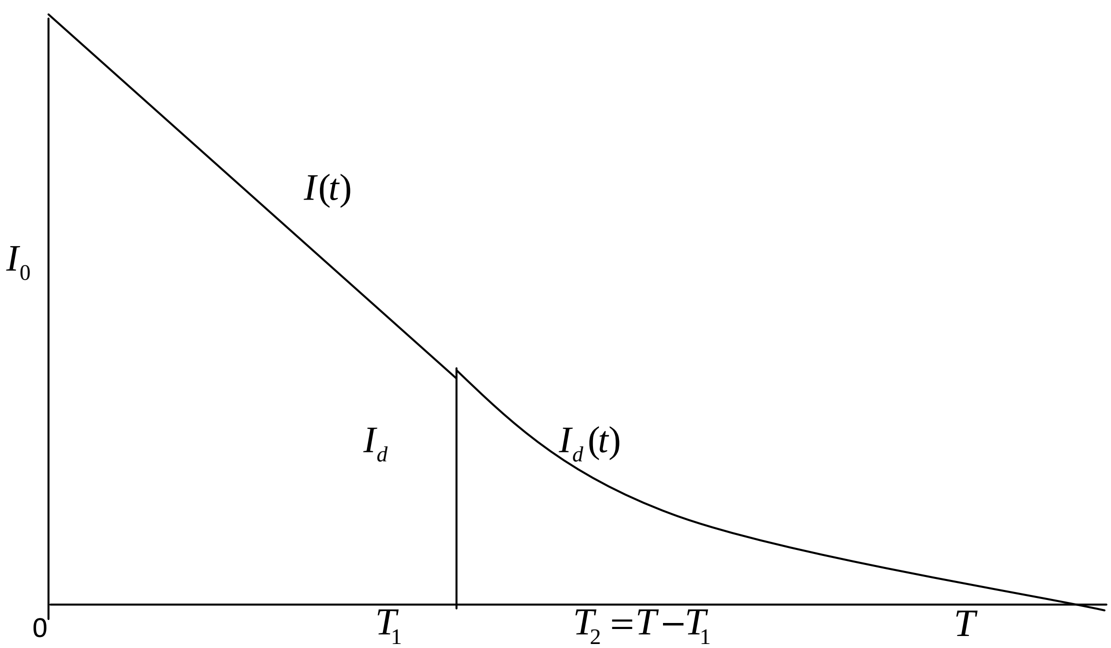

Figure I: Inventory depletion in a delayed deterioration situation

Before deterioration sets in, the depletion of inventory occurs only due to demand. This is represented by the differential equation

$$\frac{dI(t)}{dt} = -\beta_1, \qquad 0 \le t \le T_1 \tag{1}$$

Separating the variables and solving equation (1) yields:

$$I(t) = -\beta_1 t + \lambda_1 \tag{2}$$

where λ_1 is an arbitrary constant. Now, at $t=0$, $I(t) = I_0$, equation (2) becomes $I_0 = \lambda_1$, so that from (2), we get

$$I(t) = -\beta_1 t + I_0 \tag{3}$$

Moreover at $t=T_1$, $I(t) = I_d$, we obtain from (3)

$$I_0 = I_d + \beta_1 T_1 \tag{4}$$

Substituting equation (4) into equation (3), we have

$$I(t) = I_d + (T_1 - t)\beta_1 \tag{5}$$

After deterioration sets in, the inventory depletion will depend on both demand and deterioration. This is represented by the differential equation

$$\frac{dI_d(t)}{dt} + \theta I_d(t) = -\beta_2, \qquad T_1 \le t \le T \tag{6}$$

The solution of equation (6) is given by :

$$I_d(t) = -\frac{\beta_2}{\theta} + \lambda_2 e^{-\theta t} \tag{7}$$

Where λ_2 is an arbitrary constant, applying the conditions at $t = T_1,\ I_d(t) = I_d$, yields from equation (7), $I_d = -\frac{\beta_2}{\theta} + \lambda_2 e^{-\theta T_1}$

$$\therefore \quad \lambda_2 = \left(I_d + \frac{\beta_2}{\theta} \right) e^{\theta T_1} \tag{8}$$

Substituting equation (8) into (7) yields,

$$I_d(t) = -\frac{\beta_2}{\theta} + \left(I_d e^{\theta T_1} + \frac{\beta_2}{\theta} e^{\theta T_1} \right) e^{-\theta t} = -\frac{\beta_2}{\theta} + \left(I_d + \frac{\beta_2}{\theta} \right) e^{(T_1 - t)\theta}$$

$$\therefore \quad I_d(t) = \frac{\beta_2}{\theta}\left(e^{(T_1 - t)\theta} - 1\right) + I_d e^{(T_1 - t)\theta} \tag{9}$$

Now, at $t=T$, $I_d(t) = 0$, equation (9) becomes

$$0 = \frac{\beta_2}{\theta}\left(e^{(T_1 - T)\theta} - 1\right) + I_d e^{(T_1 - T)\theta} \Rightarrow I_d = -\frac{\beta_2}{\theta}\left(e^{(T_1 - T)\theta - (T_1 - T)\theta} - e^{(T - T_1)\theta}\right)$$

$$\therefore \quad I_d = \frac{-\beta_2}{\theta}\left(1 - e^{(T - T_1)\theta}\right) \tag{10}$$

Substituting equation (10) into (9) yields

$$I_d(t) = \frac{\beta_2}{\theta}\left(e^{(T_1 - t)\theta} - 1\right) - \frac{\beta_2}{\theta}\left(1 - e^{(T - T_1)\theta}\right) e^{(T_1 - t)\theta} = \frac{\beta_2}{\theta}\left(e^{(T - t)\theta} - 1\right) \tag{11}$$

Now, substituting equation (10) into (5) yields

$$I(t) = -\frac{\beta_2}{\theta}\left(1 - e^{(T - T_1)\theta}\right) + (T_1 - t)\beta_1 \tag{12}$$

4.0 Computation of the Total Inventory Costs

The total inventory is made up of the sum of the inventory ordering cost, cost due to deterioration of inventory items and the total inventory holding cost. The costs are computed individually before they are summed together.

(a) The inventory ordering cost is given as A
(b) To compute the cost due to deterioration of inventory, we note that:
The total demand between T_1 and T = the demand rate at the beginning of deterioration x the time period during which the item deteriorates.

$\therefore$ Total demand $= \beta_2 T_2 = \beta_2 (T - T_1)$

The number of items that deteriorate during the interval, $[T_1, T]$ is given as:

$$N(d_t) = I_d - \beta_2 T_2 = I_d - \beta_2 (T - T_1) \quad (13)$$

Substituting equation (10) into (13) to have

$$N(d_t) = -\frac{\beta_2}{\theta}(1 - e^{(T-T_1)\theta}) - \beta_2 (T - T_1) = -\frac{\beta_2}{\theta}(1 - e^{(T-T_1)\theta} + \theta(T - T_1)) \quad (14)$$

and the total cost due to deterioration of inventory items is given as:

$$CN(d_t) = -\frac{C\beta_2}{\theta}(1 - e^{(T-T_1)\theta} + \theta(T - T_1)) \quad (15)$$

(c) The total holding cost is given as:

$$C_H = i\int_0^{T_1} h(t)I(t)dt + i\int_{T_1}^{T} h(t)I_d(t)dt$$

Inventory Carrying Cost (or Holding Cost)

The inventory carrying cost or holding cost in a cycle is C_H. This is the cost associated with the storage of the inventory until it is sold or used. It is given as:

$$C_H = i\int_0^{T_1} h(t)I(t)dt + i\int_{T_1}^{T} h(t)I_d(t)dt$$

$$= i\int_0^{T_1} (\alpha_1 + \alpha_2 t)\left(-\frac{\beta_2}{\theta}(1 - e^{(T-T_1)\theta}) + (T_1 - t)\beta_1\right)dt + i\int_{T_1}^{T} (\alpha_1 + \alpha_2 t)\left(\frac{\beta_2}{\theta}(e^{(T-t)\theta} - 1)\right)dt$$

$$= \frac{-i\beta_2\alpha_1}{\theta}\int_0^{T_1} dt + \frac{i\beta_2\alpha_1 e^{(T-T_1)\theta}}{\theta}\int_0^{T_1} dt + i\beta_1\alpha_1 T_1\int_0^{T_1} dt - i\beta_1\alpha_1\int_0^{T_1} tdt - \frac{i\beta_2\alpha_2}{\theta}\int_0^{T_1} tdt + \frac{i\beta_2\alpha_2 e^{(T-T_1)\theta}}{\theta}\int_0^{T_1} tdt + i\beta_1\alpha_2 T_1\int_0^{T_1} tdt$$

$$- i\beta_1\alpha_2\int_0^{T_1} t^2 dt + \frac{i\beta_2\alpha_1}{\theta}\int_{T_1}^{T} e^{(T-t)\theta}dt - \frac{i\beta_2\alpha_1}{\theta}\int_{T_1}^{T} dt + \frac{i\beta_2\alpha_2}{\theta}\int_{T_1}^{T} te^{(T-t)\theta}dt - \frac{i\beta_2\alpha_2}{\theta}\int_{T_1}^{T} tdt$$

$$= \frac{-i\beta_2\alpha_1}{\theta}[t]_0^{T_1} + \frac{i\beta_2\alpha_1 e^{(T-T_1)\theta}}{\theta}[t]_0^{T_1} + i\beta_1\alpha_1 T_1[t]_0^{T_1} - \frac{i\beta_1\alpha_1}{2}[t^2]_0^{T_1} - \frac{i\beta_2\alpha_2}{2\theta}[t^2]_0^{T_1} + \frac{i\beta_2\alpha_2 e^{(T-T_1)\theta}}{2\theta}[t^2]_0^{T_1}$$

$$+ \frac{i\beta_1\alpha_2 T_1}{2}[t^2]_0^{T_1} - \frac{i\beta_1\alpha_2}{3}[t^3]_0^{T_1} - \frac{i\beta_2\alpha_1}{\theta^2}\left[e^{(T-t)\theta}\right]_{T_1}^{T} - \frac{i\beta_2\alpha_1}{\theta}[t]_{T_1}^{T} + \frac{i\beta_2\alpha_2}{\theta}\left\{-\frac{1}{\theta}\left[te^{(T-t)\theta}\right]_{T_1}^{T} - \frac{1}{\theta^2}\left[e^{(T-t)\theta}\right]_{T_1}^{T}\right\}$$

$$- \frac{i\beta_2\alpha_2}{2\theta}[t^2]_{T_1}^{T}$$

$$= \frac{-i\beta_2\alpha_1 T_1}{\theta} + \frac{i\beta_2\alpha_1 T_1 e^{(T-T_1)\theta}}{\theta} + \frac{i\beta_1\alpha_1 T_1^2}{2} - \frac{i\beta_2\alpha_2 T_1^2}{2\theta} + \frac{i\beta_2\alpha_2 T_1^2 e^{(T-T_1)\theta}}{2\theta} + \frac{i\beta_1\alpha_2 T_1^3}{2} - \frac{i\beta_1\alpha_2 T_1^3}{3}$$

$$- \frac{i\beta_2\alpha_1 e^{(T-T)\theta}}{\theta^2} + \frac{i\beta_2\alpha_1 e^{(T-T_1)\theta}}{\theta^2} - \frac{i\beta_2\alpha_1 T}{\theta} + \frac{i\beta_2\alpha_1 T_1}{\theta} + \frac{i\beta_2\alpha_2}{\theta}\left\{\frac{-Te^{(T-T)\theta}}{\theta} + \frac{T_1 e^{(T-T_1)\theta}}{\theta} - \frac{e^{(T-T)\theta}}{\theta^2} + \frac{e^{(T-T_1)\theta}}{\theta^2}\right\}$$

$$- \frac{i\beta_2\alpha_2 T^2}{2\theta} + \frac{i\beta_2\alpha_2 T_1^2}{2\theta}$$

$$= \frac{i\beta_2\alpha_1T_1e^{(T-T_1)\theta}}{\theta} + \frac{i\beta_1\alpha_1T_1^2}{2} + \frac{i\beta_2\alpha_2T_1^2e^{(T-T_1)\theta}}{2\theta} + \frac{i\beta_2\alpha_2T_1e^{(T-T_1)\theta}}{\theta^2} + \frac{i\beta_1\alpha_2T_1^3}{2} - \frac{i\beta_1\alpha_2T_1^3}{3} - \frac{i\beta_2\alpha_1}{\theta^2}$$
$$+ \frac{i\beta_2\alpha_1e^{(T-T_1)\theta}}{\theta^2} + \frac{i\beta_2\alpha_2e^{(T-T_1)\theta}}{\theta^3} - \frac{i\beta_2\alpha_1T}{\theta} - \frac{i\alpha_2\beta_2T}{\theta^2} - \frac{i\alpha_2\beta_2}{\theta^3} - \frac{i\alpha_2\beta_2T^2}{2\theta}$$

$$\therefore C_H = \left\{e^{(T-T_1)\theta} + \frac{\beta_1T_1\theta}{2\beta_2} + \frac{\alpha_2T_1e^{(T-T_1)\theta}}{2\alpha_1} + \frac{\alpha_2e^{(T-T_1)\theta}}{\theta\alpha_1} + \frac{\beta_1T_1^2\theta\alpha_2}{2\beta_2\alpha_1} - \frac{\beta_1T_1^2\theta\alpha_2}{3\beta_2\alpha_1} - \frac{1}{\theta T_1} + \frac{e^{(T-T_1)\theta}}{T_1\theta} + \frac{e^{(T-T_1)\theta}\alpha_2}{\alpha_1T_1\theta^2}\right.$$
$$\left. - \frac{T}{T_1} - \frac{\alpha_2T}{\theta\alpha_1T_1} - \frac{\alpha_2}{\theta^2\alpha_1T_1} - \frac{\alpha_2T^2}{2\alpha_1T_1}\right\}\frac{i\beta_2\alpha_1T_1}{\theta} \qquad (16)$$

4.1 Computation of the total Inventory Costs

The Total Inventory cost per unit time T is given as

$TC(T) = \frac{1}{T}$ (Inventory ordering cost + Cost due to deterioration of inventory items
+ Total inventory holding cost)

$$\therefore TC(T) = \frac{1}{T}(A + CN(d_t) + C_H)$$
$$= \frac{A}{T} - \frac{C\beta_2}{\theta T}(1 - e^{(T-T_1)\theta} + \theta(T - T_1)) + \left\{e^{(T-T_1)\theta} + \frac{\beta_1T_1\theta}{2\beta_2} + \frac{\alpha_2T_1e^{(T-T_1)\theta}}{2\alpha_1} + \frac{\alpha_2e^{(T-T_1)\theta}}{\theta\alpha_1} + \frac{\beta_1T_1^2\theta\alpha_2}{2\beta_2\alpha_1}\right.$$
$$\left. - \frac{\beta_1T_1^2\theta\alpha_2}{3\beta_2\alpha_1} - \frac{1}{\theta T_1} + \frac{e^{(T-T_1)\theta}}{T_1\theta} + \frac{e^{(T-T_1)\theta}\alpha_2}{\alpha_1T_1\theta^2} - \frac{T}{T_1} - \frac{\alpha_2T}{\theta\alpha_1T_1} - \frac{\alpha_2}{\theta^2\alpha_1T_1} - \frac{\alpha_2T^2}{2\alpha_1T_1}\right\}\frac{i\beta_2\alpha_1T_1}{\theta T} \qquad (17)$$

To determine the value of T which minimizes the total inventory costs per unit time, we differentiate equation (17) with respect to T and equate to zero to have

$$\frac{dTC(T)}{dT} = \frac{-A}{T^2} - \frac{C\beta_2}{\theta}\left\{-\frac{1}{T^2} - \frac{(T\theta-1)e^{(T-T_1)\theta}}{T^2} + \frac{\theta T_1}{T^2}\right\} + \left\{\frac{(T\theta-1)e^{(T-T_1)\theta}}{T^2} - \frac{\beta_1T_1\theta}{2\beta_2T^2} + \frac{\alpha_2T_1(T\theta-1)e^{(T-T_1)\theta}}{2\alpha_1T^2}\right.$$
$$+ \frac{\alpha_2(T\theta-1)e^{(T-T_1)\theta}}{\alpha_1\theta T^2} - \frac{\beta_1\alpha_2T_1^2\theta}{2\beta_2\alpha_1T^2} + \frac{\beta_1\alpha_2\theta T_1^2}{3\beta_2\alpha_1T^2} + \frac{1}{\theta T_1T^2} + \frac{(T\theta-1)e^{(T-T_1)\theta}}{\theta T_1T^2} + \frac{\alpha_2(T\theta-1)e^{(T-T_1)\theta}}{\alpha_1T_1\theta^2T^2}$$
$$\left. + \frac{\alpha_2}{\theta^2\alpha_1T_1T^2} - \frac{\alpha_2}{2\alpha_1T_1}\right\}\frac{i\beta_2\alpha_1T_1}{\theta} = 0 \qquad (18)$$

Multiplying equation (18) through by T^2 yields

$$-A - \frac{C\beta_2}{\theta}[-1 - (T\theta-1)e^{(T-T_1)\theta} + \theta T_1] + \left\{(T\theta-1)e^{(T-T_1)\theta} - \frac{\beta_1T_1\theta}{2\beta_2} + \frac{\alpha_2T_1(T\theta-1)e^{(T-T_1)\theta}}{2\alpha_1} + \frac{\alpha_2(T\theta-1)e^{(T-T_1)\theta}}{\alpha_1\theta}\right.$$
$$\left. - \frac{\beta_1\alpha_2T_1^2\theta}{2\beta_2\alpha_1} + \frac{\beta_1\alpha_2\theta T_1^2}{3\beta_2\alpha_1} + \frac{1}{\theta T_1} + \frac{(T\theta-1)e^{(T-T_1)\theta}}{T_1\theta} + \frac{\alpha_2(T\theta-1)e^{(T-T_1)\theta}}{\alpha_1T_1\theta^2} - \frac{\alpha_2}{\theta^2\alpha_1T_1} - \frac{\alpha_2T^2}{2\alpha_1T_1}\right\}\frac{i\beta_2\alpha_1T_1}{\theta} = 0$$

(19)

So that given other parameters we can use equation (19) to determine the best T which minimizes the total inventory cost.

5.0 Computation of the Economic Order Quantity (EOQ)

The EOQ of the corresponding best cycle length T can be routinely obtained from:

$$EOQ = \beta_1T_1 + \beta_2T_2 + N(d_t) = \beta_1T_1 + \beta_2(T - T_1) + N(d_t)$$

$$= \beta_1 T_1 + \beta_2 (T - T_1) - \frac{\beta_2}{\theta}(1 - e^{(T-T_1)\theta} + (T - T_1)\theta) \tag{20}$$

6.0 Numerical examples

Table 1 gives the solutions of seven different numerical examples having different parameters.

Table 1: Parameter values and the optimal cycle length, T for the inventory model with linear time dependent holding cost

S/N	A (N)	C	β_1 (Units)	β_2 (Units)	i	T_1	θ	α_1	α_2	T	$TC(T)$	*EOQ (Units)*
1	100	30	500	200	0.04	0.0384(14 days)	0.60	0.02	8.00	0.2328 (85 days)	734.08	99
2	150	60	600	300	0.06	0.0575(21 days)	0.50	1.00	6.00	0.1918 (70 days)	1217.12	116
3	200	100	500	300	0.07	0.0767(28 days)	0.30	-0.50	5.00	0.2219 (81 days)	1334.76	126
4	300	80	700	400	0.08	0.0959(35 days)	0.40	0.03	9.00	0.2356 (86 days)	1816.49	181
5	500	150	1000	600	0.09	0.1151(42 days)	0.20	-0.04	-11.00	0.2521 (92) days	2856.62	285
6	700	250	1500	1000	0.11	0.1343(49 days)	0.25	-0.05	6.00	0.2027 (74 days)	4182.57	339
7	1000	200	3000	1500	0.14	0.1534(56 days)	0.35	0.09	0.07	0.2082 (76 days)	5567.06	626

7.0 Conclusion

In this paper, we present a mathematical model on the inventory of deteriorating items which do not start deteriorating until after some time. The model is built on the assumption that the holding cost for the inventory item is a linear time dependent function.

The model considers a situation where the customer is expected to pay for the items as soon as they are received in the inventory, meaning that the retailer's capital is not constraint in this model.

The optimal cycle length *T* that gives the minimum total inventory cost was determined in each of the seven examples given in Table 1.

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