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A note on Markovian manpower models

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Abstract

In modelling manpower systems, most authors rely on Markov-based theoretic methodology as an analytic tool to unify the states of the system with the axiomatic foundation that there is a one-stage dependence of events. In this study, Markovian manpower models are surveyed. Specific areas are highlighted as future research directions.

Keywords: Markov process; model; manpower systems; stochastic environment.

1.0 Introduction

The descriptive and prescriptive objectives of understanding, predicting, and controlling a manpower system require the use of transition models. An operational knowledge in this area calls for Markov chain theory. A Markov chain is a discrete-state Markov process $\{X_t \mid t \in T\}$, where

$$\text{Prob}\{\text{X}_{\text{t}+1} = s_j \mid \text{X}_0 = s_1, \ldots, \text{X}_\text{t} = s_i\} = \text{Prob}\{\text{X}_{\text{t}+1} = s_j \mid \text{X}_\text{t} = s_i\},\ \text{t} = 0,1,2,\ldots \tag{1}$$

The Markov chain described by equation (1) is called a non-homogeneous discrete-time Markov chain [1]. If $\text{Prob}\{\text{X}_{\text{t}+1} = s_j \mid \text{X}_\text{t} = s_i\} = p_{ij}$, then the Markov chain is said to be homogeneous. A Markov chain with at least one absorbing state is referred to as an absorbing Markov chain. Let $\mathbf{A}$ denote an $N \times N$ transition matrix of an absorbing Markov chain with r absorbing states. Then the nonabsorbing (transient) states is $m = N - r$. Let $\mathbf{w}$ be an $m \times r$ transition matrix from nonabsorbing to absorbing states; let $\mathbf{P}$ be an $m \times m$ transition matrix among the transient states; let $\mathbf{I}$ be an $r \times r$ identity matrix; and let $\mathbf{0}$ be an $r \times m$ matrix whose entries are all zero. Then matrix $\mathbf{A}$ can be represented in the canonical form as $\mathbf{A} = \begin{bmatrix} \mathbf{P} & \vdots & \mathbf{w'} \\ \cdots & \cdots & \cdots \\ \mathbf{0} & \vdots & \mathbf{I} \end{bmatrix}$, where the prime in $\mathbf{w}$ denotes transposition. By the matrix multiplication of a partitioned matrix, we have in general that $\mathbf{A}^t = \begin{bmatrix} \mathbf{P}^t & \vdots & \mathbf{w}_t^* \\ \cdots & \cdots & \cdots \\ \mathbf{0} & \vdots & \mathbf{I} \end{bmatrix}$,

where $\mathbf{w}_t^* = \mathbf{w'} + \mathbf{Pw'} + \mathbf{P}^2\mathbf{w'} + \cdots + \mathbf{P}^{t-1}\mathbf{w'}$. If $t \to \infty$, $\lim_{t\to\infty} \mathbf{P}^t = \mathbf{0}$ since $\|\mathbf{P}\| < 1$ as $\mathbf{P}$ is sub-stochastic so that $\lim_{t\to\infty} \mathbf{w}_t^* = \mathbf{w'} + \mathbf{Pw'} + \mathbf{P}^2\mathbf{w'} + \cdots = (\mathbf{I} - \mathbf{P})^{-1}\mathbf{w'}$. The matrix $(\mathbf{I} - \mathbf{P})^{-1}$ is called the fundamental matrix of the absorbing Markov chain and its (i, j)th entry is the expected number of times the process is in a transient state i

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before entering an absorbing state j. Ibe [1] stated that a stochastic process $\{X(t)| t \geq 0\}$ is a continuous-time Markov chain if, for all $s, t \geq 0$ and non-negative integers i, j, k,

$$\text{Prob}\{X(t+s) = j|X(s) = i, X(u) = k\} = \text{Prob}\{X(t+s) = j|X(s) = i\}, \tag{2}$$

i.e. in a continuous-time Markov chain the conditional probability of the future state at time $t+s$ given the present state at s and all past states depends only on the present state and is independent of the past. If $\text{Prob}\{X(t+s) = j|X(s) = i\}$ is independent on s, then the process $\{X(t)| t \geq 0\}$ is said to be time-homogeneous. Further details on Markov chain theory can be found in [2 – 4]. For a consolidated account of early work on inference for finite Markov chains from counts of transitions, reference should be made to [5] and [6].

Markov models have been applied to diverse substantive topics. Markov models have been used as a convenient framework for analyzing the structural mechanisms which underlie social change and for extrapolating shifts in the state distribution of a population [7 – 9]. Berman and Ianovsky [10] used Markov decision approach to resolve the problem of finding the optimal switching decisions of cross-trained workers between the front and back room in a system. Bartolucci et al. [11] applied Markov model to evaluate the performance of nursing homes. Gupta et al. [12] developed a Markov model for the comparative evaluation of alternative maintenance strategies in the coal handling unit of a thermal power plant with two states, namely: working and failed states. Yu et al. [13] dealt with a learning problem of which the decision maker interacts with a standard Markov decision process where the reward functions vary arbitrarily over time; and Even-Dar et al. [14] concentrated on a Markov decision process in which the reward function is allowed to change after each time step. Litvak and Ejov [15] considered the Hamiltonian cycle problem embedded in a Markov decision process corresponding to a given graph. Leder et al. [16] applied an approximation technique for a deviation matrix of continuous-time Markov processes to queues. Horner et al. [17] studied the zero-sum Markov games. Zhang [18] studied partially observable Markov decision processes with finite state, action, observable sets, and discounted rewards. Lim and Desai [19] proposed a Markov decision process approach for a route selection problem which minimizes the cost of undesirable events. Nielsen et al. [20] consider methods for embedding a state space model into a Markov decision process with particular reference to agriculture.

2.0 Estimation of transition probabilities of Markov chains

Anderson and Goodman [5] proposed the maximum likelihood method for estimating the transition probabilities of a Markov chain. Billingsley [21] maintained that in estimating the transition probabilities of a Markov chain, there arises a natural problem of testing whether the transition probabilities have a specified form $p_{ij}(\theta)$ say, where θ is an unknown parameter vector which is estimated from the sample. Billingsley [21] therefore stated that if the process is governed by the transition matrix $\{p_{ij}(\theta)\}$, the log-likelihood of the observation $\{x_1, \ldots, x_{n+1}\}$ is $\sum_{ij} f_{ij} \log p_{ij}(\theta)$; and if the parameter is a vector $\theta = (\theta_1, \ldots, \theta_r)$ with r real components, then a solution $\hat{\theta}$ is obtained by solving the system of the maximum likelihood equations $\sum_{ij} \frac{f_{ij}}{p_{ij}(\theta)} \frac{\partial p_{ij}(\theta)}{\partial \theta_u} = 0, \quad u = 1, \ldots, r$. Since Anderson and Goodman [5] and Billingsley [21] proposed the maximum likelihood method for estimating the transition probabilities of a Markov chain, there have been further developments in this aspect. Raghavendra [22] estimated the transition probabilities of a Markov manpower model using a bivariate probability distribution, which consist of seniority and performance rating, for framing promotional policies. The limitation of the method is that the calculated values of the transition probabilities cannot be guaranteed to be in the range $(0, 1)$. Kulperger and Prakasa Rao [23] proposed the bootstrap technique as a method for obtaining approximation to the sampling distributions from which the estimates of transition probabilities can be calculated. One of the drawbacks in the application of the bootstrap technique is the computational agony in obtaining a large bootstrap replicate from which such sampling distributions can be approximated. More so, Davis et al. [24] presented a method in which the transition probabilities of the Markov process is estimated from aggregate data using the logarithms of partial odds. Partial odds refer to the quotient of the probability of a transition to another state to the probability of no transition. Authors such as [25 – 27], have proposed several statistics for testing the assumption of homogeneity about the transition probabilities. Sales [26] posited that the statistic of goodness of fit given as:

$$\sum \frac{(\text{observed values} - \text{expected values})^2}{\text{expected values}}, \tag{3}$$

is not applicable to manpower systems because the observed values are not independent. Zanakis and Maret [28] posited that the entire transition probability matrix is constant over time if

$$\sum_{i=1}^{s}\sum_{j=1}^{m}\sum_{t=1}^{T}\frac{n_i(t)[\hat{p}_{ij}(t)-\hat{p}_{ij}]^2}{\hat{p}_{ij}} < \chi^2_{\alpha}[s(m-1)(T-1)], \qquad (4)$$

where s is the number of non-absorbing states, m is the total number of states, T is the number of time periods, $n_i(t)$ is the number of persons in state i during period t; $\hat{p}_{ij}(t)$ is the estimated transition probability from state i to j during period t; $\hat{p}_{ij}$ is the pooled estimate of $\hat{p}_{ij}(t)$; and $\chi^2_{\alpha}[s(m-1)(T-1)]$ is the chi-square value at α percentile and $[s(m-1)(T-1)]$ degrees of freedom.

3.0 Manpower modelling based on Markov processes

Bartholomew et al. [29] stipulated that manpower planning is an attempt to match the supply of people with the jobs available for them. Anthony and Wilson [30] described manpower modelling as taking a group of people who are similar in some way (i.e. all working in the same organization, all enrolled in the same school, etc.) and subdivide them into homogeneous groupings (e.g. by age, level, etc.) so that by using historical data and intuition one can examine future possibilities and the effects of organizational policies. The flow of people in manpower systems is subdivided into recruitment stream, the transition between states and wastage from the system. Setlhare [31] studied the optimization and estimation of stochastic model of manpower systems with particular reference to Markov processes. Hopkins [32] discussed the use of Markov processes in staff planning and evaluation under certain organizational policies. Several hierarchical models based on Markov chain theory have been applied to manpower systems [33 – 38]. Smith and Bartholomew [39] wrote a comprehensive chronicles on manpower planning in the United Kingdom. In the work they credited the use of Markov chain theory to model graded systems of known total size to the paper jointly published by A. Young and G. Almond in 1961. Bartholomew et al. [29] discussed the use of Markov chains as a convenient framework for analyzing the state-transitions in graded systems. Some authors [40, 41] concentrated on the deterministic aspects of Markov chains and therefore coined the term ‘fractional-flow models’ for Markov-based transition models. Purkiss [42] explored papers on the practical relevance of Markovian manpower models (MMMs), while Edwards [43] focused on the assumptions on MMMs and its applications. Since the survey by Purkiss [42] and Edwards [43], over seventeen years have elapsed. In these intervening years, MMMs have been firmly established in literature, especially in the evaluation of the educational process [9, 44, 45].

3.1 Transition models for manpower systems based on Markov chains

Social processes such as educational and manpower systems are modelled either as a closed system or an open system using Markov chain theory [46 – 50]. In a closed system with a set of states $S=\{1,\ 2,\dots,k\}$, where no attrition is allowed, the transition probability matrix $\mathbf{P}\in \boldsymbol{R}^{k^2}$ is a stochastic matrix given as $\mathbf{P}=\left\{(p_{ij}):\sum_{j=1}^{k}p_{ij}=1,\ p_{ij}\geq 0,\ i,j\in S\right\}$, where $\boldsymbol{R}^{k^2}$ is the k^2-dimensional Euclidean space and p_{ij} is the transition probability from state i to state j. The distribution of the closed system at the next time point is computed as [51]:

$$\bar{\mathbf{n}}(t+1)=\mathbf{n}(t)\mathbf{P}, \qquad (5)$$

where $\mathbf{n}(t)=(n_1(t),\ n_2(t),\ \cdots,n_k(t))$ is a point in the $k-$dimensional Euclidean space with $n_i(t)$ being the number of individuals in state $i\in S$ in period t, and $\bar{\mathbf{n}}(t+1)$ is the expected manpower structure in period $t+1$.

On the other hand, there is a two-way flow between the system and the outside world in an open system. Thus, the matrix $\mathbf{P}$ is sub-stochastic, i.e., $\mathbf{P}=\left\{(p_{ij}):\sum_{j=1}^{k}p_{ij}\leq 1, p_{ij}\geq 0,\ i,j\in S\right\}$, where the shortfall in the sum of transition probabilities is created by wastage in the system. The transition model for the open system is modelled as [52]:

$$\bar{\mathbf{n}}(t+1) = \mathbf{n}(t)\mathbf{P} + R(t)\mathbf{r}, \qquad (6)$$

where $R(t)$ is the total recruitment at period t, and $\mathbf{r}$ and $\mathbf{w}$ denote the probability vectors of recruits and wastage, respectively. By assuming that recruitment is done to achieve desired growth, g, and to replace leavers [53], we have

$$\bar{\mathbf{n}}(t+1) = \mathbf{n}(t)\mathbf{Q} + g\mathbf{n}(t)\mathbf{Ie'r}, \qquad (7)$$

where $\mathbf{Q} = (\mathbf{P} + \mathbf{w'r})$ and $\mathbf{e'}$ is a column vector of ones conformable with matrix $\mathbf{Q}$. The matrix $\mathbf{Q}$ is stochastic since $\mathbf{Qe'} = \mathbf{e'}$. In this study, the prime notation $(')$ denotes transposition. The use of Markov-based models for open manpower systems has received renewed interests as a means to a better quantitative planning [54].

Looking at the present state of literature, we see that Markovian manpower models (MMMs) are either formulated in discrete-time [55 – 59] or in continuous-time [60 – 63]. Some authors in dealing with manpower systems assumed that the individual transition between states take place according to a homogeneous Markov chain [26, 53], while others [64 – 66] assumed a non-homogeneous Markov model for manpower systems. The use of the theory of non-homogeneous Markov systems (NHMS) has flourished in literature on manpower systems [67 – 69]. Vassiliou and Tsaklidis [67] examined the cyclic behaviour of NHMS. In such a system with period d, the transition probabilities satisfy the equations:

$\mathbf{P}(nd+r) = \mathbf{P}(r)$, $\mathbf{P}_0(nd+r) = \mathbf{P}_0(r)$, and $\mathbf{P}_{k+1}(nd+r) = \mathbf{P}_{k+1}(r)$, so that $\mathbf{Q}(nd+r) = \mathbf{Q}(r)$, $n = 1,2,\cdots$, and $r = 0,1,2,\cdots,d-1$.

Tsantas and Vassiliou [69] introduced the concept of a non-homogeneous Markov system in a stochastic environment. In the work, a sequence of stochastic matrices $\{\mathbf{C}(t)\}_{t=1}^{\infty}$ which is the outcome of the choice of a strategy under various pressures in the environment was employed. The sequence of stochastic matrices is possible since there is no specific transition matrix in a stochastic environment. The sequence $\{\mathbf{C}(t)\}_{t=1}^{\infty}$ is called the compromised non-homogeneous Markov chain. The compromised non-homogeneous Markov chain is applicable when in the previous sequence of a system a repetition of a particular number of matrices is observed without a certain deterministic pattern. Vassiliou and Georgiou [70] analyzed the asymptotically attainable structures of the embedded non-homogeneous Markov chain defined by the sequence $\{\mathbf{Q}(t)\}_{t=0}^{\infty}$, where $\mathbf{Q}(t) = \mathbf{P}(t) + \mathbf{P}'_{k+1}(t)\mathbf{P}_0(t)$. The matrix $\mathbf{P}(t)$ is a transition probability matrix at time t; $\mathbf{P}_{k+1}(t)$ is a $1\times k$ vector of loss probabilities at time t; and $\mathbf{P}_0(t)$ is a $1\times k$ vector of input probabilities at time t. Later on, Tsaklidis [71] studied the evolution of attainable structures of a homogeneous Markov system (HMS) with fixed size, where the transition matrix $\mathbf{Q} = \mathbf{P} + \mathbf{P}_{n+1}\mathbf{P}'_0$ was reffered to as the imbedded Markov chain of the HMS. Here, $\mathbf{P}$ is a transition probability matrix; $\mathbf{P}_{n+1}$ denotes an $n\times 1$ loss vector; and $\mathbf{P}_0$ is an $n\times 1$ recruitment probability vector. The block structure of matrices $\mathbf{Q}(t)$ and $\mathbf{Q}$ respectively in [70] and [71] is analogous to the Young/Almond-type Markov chain formulation [46].

3.2 Semi-Markov models

The basic Markov transition model contains some fundamental drawbacks. These include, *inter alia*, the assumption that transitions to another grade or leaving occur at a constant rate and that the conditional probability of moving having survived to time t does not depend on t. Semi-Markov models account for variations in leaving probability with duration and destination. The semi-Markov model extends the simple Markov transition model to a formulation which combines a transition matrix of probabilities of moving between grades with a conditional distribution of duration in a grade before transition to each destination. Mehlmann [72] had proved the asymptotic relation for the manpower structure of a semi-Markov manpower system with Poisson input. Vassiliou and Papadopoulou [73] studied the maintainability of expected duration structure in the state of a system using the non-homogeneous semi-Markov system. McClean [74] studied a semi-Markov model for a multigrade population with Poisson recruitment. The basic problem encountered in the estimation of the parameters of semi-Markov models was identified by McClean [75] as incompleteness in the data. The reason for this is that the available data are either left truncated or right censored. McClean [75] described left truncated and right censored data as follow: suppose an individual is first observed in a particular state Z_0 and last observed in the state Z_m. Then the complete history of the individual may be described by

$H = \{T_0, Z_0, T_1, Z_1, \ldots, Z_{m-1}, T_m, Z_m\}$, where for left truncated data, $T_0 > 0$ and represents the time already spent in the system when data collection commences. A time T_1 is subsequently spent in Z_0 before transfer to Z_1. For right censored data $Z_m = k+1$ and the data are right censored at T_m.

3.3 Hidden Markov models

A hidden Markov model (HMM) is a stochastic process whose evolution is governed by an underlying discrete Markov process (Markov chain) with a finite number of states $s_i \in S$, $i = 1, \ldots, k$, which are not directly observable. Baum and Petrie [6] introduced the concept of HMMs as a tool for probabilistic sequence modelling. Messina and Toscani [76] posited that a discrete HMM is characterized by five elements, namely: a set of states $s_i \in S$, $i = 1, \ldots, N$; a set of observable symbols $O_m \in M$, $m = 1, \ldots, k$; a probability distribution for the initial state $\pi(q_1 = s_i)$, $s_i \in S$, $i = 1, \ldots, N$; a set of state transition probabilities that can be represented by a transition matrix A, with $a_{ij} = \Pr(q_{t+1} = s_j | q_t = s_i)$ and $\sum_j a_{ij} = 1$ for each $i = 1, \ldots, N$; and a set of probability density functions B_i whose elements $b_{ik} = \Pr(k | q_t = s_i)$ give the probability of observing in state s_i the emission of the symbol $k \in M$. Messina and Toscani [76] therefore stated that a HMM is fully specified once we know the initial state distribution $\pi(s_i)$, $i = 1, \ldots, N$, the state transition probabilities in A and the output probabilities $B = [B_1, \ldots, B_N]$. Mitrophanov et al. [77] derived a tight perturbation bound for HMMs (i.e. a bound which cannot be improved by a constant factor) and proved that the distributions of a HMM show a weaker dependence on the transition probabilities than on the emission probabilities. The statistical implication of the weaker dependence of the distributions of HMM is that the transition probabilities may be more difficult to estimate. Estimation of parameters in HMMs has been theoretically discussed in [78]. A well-established method for HMM parameter estimation which is the expectation-maximization (EM) algorithm had earlier been developed [6]. Bartolucci et al. [11] applied the EM algorithm for HMM to obtain the maximum likelihood estimates of the latent Markov model which was used to evaluate nursing homes. HMMs have been used to analyze alcoholism treatment [79] and to assess chromosomal alteration [80]. However, in the available literature, none of the authors dealt with the application of HMMs to manpower systems.

4.0 Some perturbing issues in MMMs and their remedy

The primary objective in manpower planning is to predict future stocks. In doing this, there are some limitations. The limitations include: incorrectness or inapplicability of the model, estimation of parameters, random variation in the number of losses and transfers, and random variation in the input. Bartholomew [27] suggested the following methods as a means of addressing these limitations as follows: the use of cross-validation to minimize error due to incorrectness, the use of sensitivity analysis to see how much predictions are affected by changes in the transition probabilities, and the Bayesian treatment and the frequentist approach to investigate estimation errors.

Another paramount issue in modelling manpower systems is how to account for the presence of heterogeneity. The sources of heterogeneity in manpower systems include: observable sources such as length of service, marital status, sex, level of employment, time factor; and latent sources such as individual traits and environmental factors. Ugwuowo and McClean [81] reviewed several suggestions made by previous authors on ways of tackling heterogeneity in manpower planning, and thereafter proposed some techniques which include semi-Markov model, proportional hazard model, and non-time-homogeneous model, to deal with heterogeneity in manpower systems.

The implementation of Markovian manpower models (MMMs) is characterized by computational complexities. For this reason, several software packages such as KENT, PROSPECT, MICROPROSPECT, CAMPLAN, and MANSIM [29, 39] and FORMASY [84] have been developed to facilitate the use of MMMs. These packages are tailored towards specialized problems and thus cannot be easily modified. Anthony and Wilson [30] employed the spreadsheet systems in analyzing manpower systems; while in Ekhosuehi and Osagiede [82] the Matlab package was used as a computational tool in analyzing manpower systems. The Matlab package has several computational and graphic visual advantages arising from its interactive interface and independent plotting devices.

Skulj et al. [85] applied Markov chain model to the manpower structure of Slovenian Army. In this work, the problem of achieving the desired manpower structure could not be resolved. In view of this, Skulj et al. suggest the development of a semi-Markov model in which the age of the units in the segments is considered in the calculations of transition probabilities as a way of improving on the Markov chain model.

5.0 The limiting relation of MMMs

Limiting behaviour of a manpower system may relate to either expected numbers in the grade or the relative numbers and may be formulated in discrete or continuous time. Vassiliou [66] showed the conditions under which a limiting or a relative limiting structure exists for certain discrete time manpower systems. The model used in [66] is the time-dependent Markov model proposed by Young and Vassiliou [64]. Mehlmann [58] and Vassiliou [86] had earlier studied the limiting behaviour of a manpower system with Poisson recruitment and observed that the members in the various grades are asymptotically mutually independent Poisson. Vassiliou [86] found that the asymptotic age-distribution of a manpower system when the transition matrix is a general stochastic matrix (not necessarily proper) is not in general a single distribution but a repeating cycle of distributions which depends not only on the transition probabilities but also on the initial distribution. Research has also been on to determine 'how fast' the limiting relation $\lim_{t\to\infty}\mathbf{P}^t = \boldsymbol{\pi}$ converge, where $\mathbf{P} = \{p_{ij}\}$ and $\boldsymbol{\pi} = (\pi_1, \ldots, \pi_k)$. For instance, Ledermann [51] applied the metric $\mu(\mathbf{P}^t - \boldsymbol{\pi}) = \max\left(\left|p_{ij} - \pi_j\right|; 1 \le i, j \le k\right)$ and defined the speed of convergence as $\mu(\mathbf{P}^t - \mathbf{L}) \le c\alpha^t t^{k-1}$, where c is a constant that does not depend on t, and $\alpha = \max_{2\le r\le k}\sum_{r=2}^{k}\left|p_{ir} - p_{1r}\right|$.

60 The problem of control of manpower systems

Control problems have two aspects, namely: attainability and maintainability. Attainability is concerned with whether or not a desirable structure can be reached from a given structure under the constraints on the variables, while maintainability is concerned with how to remain at the desirable structure once it has been reached. The theory of maintainability aims to show what kinds of structure can be maintained under various assumptions on promotion and recruitment. In practice, the attainability problem cannot be completely solved because not all transitions can be controlled, while those that can be controlled are not allowed to be set entirely arbitrarily. In the case of maintainability, the aim is to find a transition matrix within an allowable set of matrices whose stationary distribution is the required distribution. The control of asymptotic variability of expectations, variances and covariances in a Markov chain model has become a surge of interest in manpower systems. Various works [68, 70, 87] provide significant contributions in this area. Attainable and maintainable structures in Markov manpower systems under recruitment control abound in literature. Bartholomew et al. [29] concentrated on simple arithmetic tests of whether or not a given structure can be maintained. Davies [48] considered a structure that can be maintained under a partially-stochastic model in which wastage flow is the only stochastic variable, while promotion is deterministic. Davies [48] noted that, although an initial structure $\mathbf{n}$ can be calculated from an attainable structure $\mathbf{n}^*$, there is no guarantee that all of the entries in $\mathbf{n}^*\mathbf{P}^{-1}$ for $\mathbf{P} = \{p_{ij}\}$, $|\mathbf{P}| \ne 0$, will be non-negative. Davies [48] therefore introduced the concept of a T-step path in attaining a structure $\mathbf{n}(T)$, which is the path with the maximum probability over all possible attainable T-step paths. By this concept, Davies [88] stated and proved several theorems. Later Kalamatianou [89] analyzed attainable and maintainable structures in Markov manpower systems with pressure in grades. In such a system, the Markov assumptions of constant transition rates do not hold. Tsantas and Vassiliou [69] introduced the concept of a non-homogeneous Markov system in a stochastic environment (S-NHMS). Afterwards, Tsantas and Georgiou [90] extended the concept of S-NHMS to the partial maintainability and control of a hierarchical system. Haigh [91] studied the stability of manpower systems. In [91], the case of a strictly maintainable structure, $\mathbf{xP} \le \mathbf{x}$, was considered, where $\mathbf{x}$ is a vector representing the organizational structure and $\mathbf{P} = \{p_{ij}\}$ is the transition probability matrix. Vassiliou [92] analyzed the stability of a non-homogeneous Markov chain model in manpower systems and stated several theorems in order to achieve stability in the Young/Vassiliou-type model [64].

7.0 Conclusion

The use of MMMs as a tool for evaluating manpower systems has been appraised in this paper. The major accomplishments of the study are that a large number of literature on manpower planning have been surveyed and, arising thereof, are challenges which are potential future directions of research. Nonetheless, a critical problem encountered in the use of MMMs is that the basic data requirements to estimate the transition parameters are often statistically incomplete. For instance, semi-Markov models which require data such as: stock data on the number of staff in each grade at the beginning and end of the year, and their dates of appointment to the grade, as well as data on the movers and leavers for each grade during the year, together with their dates of appointment to the grade, may not all be available. For this reason, it is necessary to have a comprehensive database in every organisation. All the same, there are some gaps to be filled in the application of MMMs. As earlier pointed out in [29], it is necessary to develop detailed dynamic models to extrapolate shifts in the structure of manpower systems. Even so, the application of hidden Markov models (HMMs) to manpower systems so as to tackle the problem posed by heterogeneity in the system is yet to be achieved.

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