Journal of the Nigerian Association of Mathematical Physics
Volume **20 (March, 2012), pp 205 – 212**

The Tomonaga Model

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Abstract

The Hamiltonian of the one-dimensional electron gas is not exactly solvable. Tomonaga upon recognizing that the excitations of the electron gas are approximately bosons, although the elementary particles, the electrons, are fermions, went on to assume that the excitations are exactly bosons. The resulting model, the Tomonaga model immediately becomes exactly solvable. This preliminary work reviews briefly what has been done so far with the Tomonaga model; details the Tomonaga model and all the equations it entails, as a starting point for subsequent studies. There is the likelihood that the model may, in the not too distant future, become as important as the Hubbard model.

1.0 Introduction

The Tomonaga model proposed by Sin-itiro Tomonaga in June 29, 1950 describes a one-dimensional electron gas. The model examines the Hamiltonian of the one-dimensional electron gas and makes some approximations on it, so as to render the resulting Hamiltonian exactly solvable.

This model has been useful in several kinds of problems. First, there are organic solids such as tetrathiofulvalinium-tetracyano-quinodimethane (TTF-TCNQ) whose conductivity is thought to be largely one-dimensional [1]. The Tomonaga model has played a role in the interpretation of electrical conductivity in these materials [2]. Secondly, in impurity problems, or X-ray absorption problems, the response of the electron gas to the central impulse can be factored into spherical harmonics associated with different angular momentum states *l*. Each angular momentum channel *l* then becomes a one-dimensional electron gas to which one may apply the Tomonaga model. Recently, semiconductor nanotechnology permits the construction of semiconductor channels which acts as one-dimensional conductors. The Tomonaga model is used in the theory of these systems. Single wall carbon nanotubes are other examples of one-dimensional conductors.

Tomonaga presented the exact solution for the long-wavelength density response of a system of interacting fermions in one spatial dimension. In order to

simplify the problem; he studied the high-density limit where the range of the interaction is much larger than the inter-particle distance [3].

The effects of the Coulomb interaction of electrons in metals can only be described approximately. With the progress in producing artificial low dimensional structures the theoretical work on one-dimensional interacting fermions has gained importance. Special features of the spectrum of low energy excitations in one dimension allow exact solutions of models of interacting fermions. The main idea can already be understood by working with noninteracting fermions, which have the same spectrum of excitations as a harmonic chain. The method of bosonization is the key concept to understand ground state properties and the spectrum of excited states with low excitation energy also for interacting fermions [4].

The homogeneous electron liquid (HEL), continues to be of interest both in three dimensions (3D) [5] and in systems of reduced dimensionality. Unlike their higher-dimensional counterparts, 1D interacting fermion systems are not Fermi liquids; instead they exhibit the low-energy phenomenology common to many 1D fermion systems, often referred to as Tomonaga-Luttinger (TL) liquids [6].

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Any TL liquid is completely specified by four parameters: the charge and spin collective excitation velocities v_ρ and v_σ, and the correlation exponents k_ρ and k_σ. In the absence of interactions v_ρ and v_σ reduce to the Fermi velocity, v_F, and the correlation exponents are equal to 1 [7].

The Tomonaga model, like the Hubbard model, is likely to enjoy a lot of applications in the future. In view of this, this work will describe in detail the Tomonaga model and all the equations it entails. Some concrete applications of the model will be discussed in a subsequent report.

2.0 Methodology

The original model of Tomonaga (1950) discusses the following Hamiltonian for the one- dimensional interacting electron gas [8]:

$$H = v_F \sum_{ks} |k| a_{ks}^{+} a_{ks} + \frac{1}{2L} \sum_{k} V_k \rho_{(k)} \rho_{(-k)} \tag{2.1}$$

$$\rho(k) = \sum_{ps} a^{+}_{p-\frac{k}{2},s} a_{p+k/2,s} \tag{2.2}$$

Where L is length of the one-dimensional system, v_F is Fermi velocity of the particles which are assumed to have a linear dispersion relation, $s = \pm 1$, is the spin, k is the possible electron states, $\rho(k)$ is electron density operator, V_k is electron-electron interaction term and a_{ks}^{+} and a_{ks} are the creation and destruction operators of electrons in site k and spin s respectively.

The basic step in the Tomonaga model is to divide the density operator into two terms, $\rho_1(k)$ and $\rho_2(k)$, with

$$\rho_1(k) = \sum_{p>o,s} a^{+}_{p-\frac{k}{2},s} a_{p+\frac{k}{2},s} \tag{2.3}$$

$$\rho_2(k) = \sum_{p<o,s} a^{+}_{p-\frac{k}{2},s} a_{p+\frac{k}{2},s} \tag{2.4}$$

$$\therefore \rho(k) = \rho_1(k) + \rho_2(k) \tag{2.5}$$

The density operator commutes with any other density operator $\rho(k)$.

We now examine the commutation relations [9]:

$$[\rho_1(k), \rho_1(k')] = \sum_{s,s'} \sum_{p,p'>0} \left[a^{+}_{p-\frac{k}{2},s} a_{p+\frac{k}{2},s} \;,\; a^{+}_{p'-\frac{k'}{2},s'} a_{p'+\frac{k'}{2},s'} \right] \tag{2.6}$$

$$[\rho_1(k), \rho_1(k')] = \sum_{s,s'} \sum_{p,p'>0} \left\{ a^{+}_{p-\frac{k}{2},s} a_{p\frac{k}{2},,s} a^{+}_{p'-\frac{k'}{s},s'} a_{p'-\frac{k'}{s},s'} - a^{+}_{p'-\frac{k'}{s},s'} a_{p'+\frac{k'}{s},s'} a^{+}_{p-\frac{k}{2},s} a_{p+\frac{k}{2},s} \right\} \tag{2.7}$$

Summing over s' and p', we obtain

$$[\rho_1(k), \rho_1(k')] = \sum_{s,p>0} \left\{ a^{+}_{p-\frac{k}{2},s} a_{p+k'+\frac{k}{2},s} \Theta\left(p + \frac{k}{2} + \frac{k'}{2} \right) - a^{+}_{p-k'-\frac{k}{2},s} a_{p+\frac{k}{2},s} \Theta\left(p - \frac{k}{2} - \frac{k'}{2} \right) \right\} \tag{2.8}$$

where, $\Theta\left(p + \frac{k}{2} + \frac{k'}{2} \right)$ is a step function, defined as

$$\left.\begin{aligned} \Theta\left(p + \frac{k}{2} + \frac{k'}{2} \right) &= 1 \quad \text{if} \quad \left| p + \frac{k}{2} + \frac{k'}{2} \right| > 0 \\ &= 0 \quad \text{if} \quad \left| p + \frac{k}{2} + \frac{k'}{2} \right| < 0 \\ &= ½ \quad \text{if} \quad \left| p + \frac{k}{2} + \frac{k'}{2} \right| = 0 \end{aligned}\right\} \tag{2.9}$$

An important special case is $k' = -k$
Then, equation (2.8) becomes

$$[\rho_1(k), \rho_1(-k)] = \sum_s \sum_{-\frac{k}{2} \le p \le \frac{k}{2}} n_{p,s} \tag{2.10}$$

where $n_{p-\frac{k}{2},s} = a^+_{p-\frac{k}{2},s} a_{p-\frac{k}{2},s}$ *and* $n_{p+\frac{k}{2},s} = a^+_{p+\frac{k}{2},s} a_{p+\frac{k}{2},s}$ (2.11)

The operator $n_{p,s}$ is replaced by its average in the ground state of the free-particle system

$$\sum_s \sum_{-\frac{k}{2} \le p \le \frac{k}{2}} n_{p,s} = 2 \sum_{-\frac{k}{2} \le p \le \frac{k}{2}} \Theta(k_f - |p|) = \begin{cases} 2\left(\frac{kl}{2\pi}\right), & k < 2k_f \\ \frac{2k_f l}{\pi}, & k > 2k_f \end{cases} \tag{2.12}$$

The commutation relation (2.10) can be written for $k < 2k_F$ as

$$[\rho_1(k), \rho_1(-k)] = \frac{kl}{\pi} \tag{2.13}$$

$$[\rho_2(k), \rho_2(-k)] = -\left(\frac{kl}{\pi}\right) \tag{2.14}$$

$$[\rho_1(k), \rho_2(-k)] = 0 \tag{2.15}$$

The analogous results are included for the other commutators, (2.14) and (2.15), which can be derived in the same way as shown above. The Tomanaga model assumes that those density operators obey the exact commutation relations of

$$\left.\begin{aligned} [\rho_1(k), \rho_1(-k')] &= \delta_{k,k'}\left(\frac{kl}{\pi}\right) \\ [\rho_2(k), \rho_2(-k')] &= -\delta_{k,k'}\left(\frac{kl}{\pi}\right) \\ [\rho_1(k), \rho_2(-k')] &= 0 \end{aligned}\right\} \tag{2.16}$$

The relations (2.16) constitute the ***central approximation*** of the Tomonaga model. The commutation relations are not exact, since the commutators give operators, as in (2.10). However, exact results are obtained when taking the expectation value of the commutation relations [10] as shown in eqn.(2.17)

$$\langle [\rho_1(k), \rho_1(-k')] \rangle = \delta_{kk'} \sum_{s,p>0} \left[\left\langle n_{p-\frac{k}{2}} \right\rangle - \left\langle n_{p+\frac{k}{2}} \right\rangle \right]$$

$$= 2\,\delta_{kk'} \sum_{-\frac{k}{2} \le p \le \frac{k}{2}} \langle n_{p,s} \rangle \tag{2.17}$$

It is convenient to express the density operators $\rho_j(\pm k)$ in terms of creation and destruction operators for bosons. These definitions are given in (2.18), where the symbol k is always positive:

$$\left.\begin{aligned} \rho_1(k) &= b_k \sqrt{\frac{kl}{\pi}} \\ \rho_1(-k) &= b_k^+ \sqrt{\frac{kl}{\pi}} \\ \rho_2(k) &= b_{-k}^+ \sqrt{\frac{kl}{\pi}} \\ \rho_2(-k) &= b_{-k} \sqrt{\frac{kl}{\pi}} \end{aligned}\right\} \tag{2.18}$$

$$\left[b_k, b_{k'}^{+}\right] = \delta_{k,k'} \tag{2.19}$$

The operators ρ_1 always commute with ρ_2. The choice (2.18) satisfies the approximate commutation relation (2.16).

The 2nd term in the Hamiltonian (2.1) may be written in terms of these boson operators:

$$\left.\begin{aligned} \rho\ (k) &= \rho_1(k) + \rho_2(k) = (b_k + b_{-k}^{+})\sqrt{\frac{kl}{\pi}} \\ \rho\ (-k) &= \rho_1(-k) + \rho_2(-k) = (b_k^{+} + b_{-k})\sqrt{\frac{kl}{\pi}} \end{aligned}\right\} \tag{2.20}$$

On substituting (2.20) into the 2nd term of (2.1), we obtain

$$\frac{1}{2L}\sum_k V_k \rho(k)\rho(-k) = \sum_k \overline{V}_k \left(b_k + b_{-k}^{+}\right)\left(b^{+} + b_{-k}\right) \tag{2.21}$$

where, $\overline{V}_k = \dfrac{V_k |k|}{2\pi}$ (2.22)

The electron-electron interaction term has been recast into an interaction between the boson excitation of the electron gas.

The 1st term in the Hamiltonian (2.1) is the particle kinetic energy. We shall now express it in terms of boson coordinates.

Let H_o be the kinetic energy term,

$$H_0 = \nu_F \sum_{k's'} |k'|\, a_{k's'}^{+} a_{k's'} \tag{2.23}$$

Its commutator with $\rho_1(k)$ is

$$\left[\rho_1(k), H_0\right] = \nu_F K \rho_1(k) \tag{2.24}$$

Substituting (2.18) into (2.24) and simplifying, we obtain

$$\left[b_k, H_0\right] = \nu_F K b_k = w_k b_k \tag{2.25}$$

where, $w_k = \nu_F k$

Next, considering the commutator of H_0 with ρ_2, we obtain

$$\left[\rho_2(k), H_0\right] = -\, w_k\ \rho_2(k) \tag{2.26}$$

Both of these approximate commutators are satisfied with the following choice for H_o

$$H_0 = \sum_k w_k b_k^{+} b_k \tag{2.27}$$

$$H = \sum_k \left[w_k b_k^{+} b_k + \overline{V}_k \left(b_k + b_{-k}^{+}\right)\left(b_k^{+} + b_{-k}\right)\right] \tag{2.28}$$

The one-dimensional electron gas (2.1) has been recast into the boson Hamiltonian (2.28), which is exactly solvable [11].

We now solve equation (2.28) exactly by changing to a coordinate representation for the boson operators:

$$Q_k = \frac{1}{\sqrt{2w_k}}\left(b_k + b_{-k}^{+}\right) \tag{2.29}$$

$$P_k = i\sqrt{w_k / 2}\left(b_k - b_{-k}^{+}\right) \tag{2.30}$$

$$\left[Q_k, p_{k'}\right] = i\delta_{k.k'} \tag{2.31}$$

In this representation the Hamiltonian is written as

$$H_0 = \frac{1}{2}\sum_w \left(P_{-k}P_k + w_k^2 Q_k Q_{-k}\right) \tag{2.32}$$

$$H = \frac{1}{2}\sum_w \left(P_{-k}P_k + E_k^2 Q_k Q_{-k}\right) \tag{2.33}$$

$$E_k^2 = w_k^2 + 4w_k\overline{V}_k \tag{2.34}$$

The new eigenfrequencies are E_k.

The Hamiltonian of the one dimensional electron gas (2.1) has been solved approximately. So far the form of interaction potential V_k has not been specified. In fact, physicists choose a variety of forms for this interaction to suit their problem. One possible choice is to take $V_k \propto e^2 = \text{constant} = V_0$, which gives the energy spectrum as

$$E_k = \overline{v}_F\ K \tag{2.39}$$

where, $\overline{v}_F = \sqrt{v_F\left(v_F + \frac{2}{\pi}V_0\right)}$ (2.40)

Another possible choice is to take $V_k = \frac{2}{3}\left(\frac{e^2k_F^2}{k^2}\right)$, and the energy spectrum is

$$E_k = \sqrt{k^2v_F^2 + w_p^2} \tag{2.41}$$

$$w_p^2 = 4w_k\overline{V}_k = \frac{4\pi e^2 n_0}{m} \tag{2.42}$$

where, $n_0 = \frac{K_F^3}{3\pi^2}$

3.0 Discussion of Results

The simplified commutation relations (2.13) to (2.26) tell us that the density field can be regarded as a Bose-field. We first consider the case of the ideal Fermi gas in which there are no interactions between particles. If the gas is not excited too highly, only particles in the neighbourhood of the Fermi maximum are raised to higher levels. There exist holes and excited particles only in neighbourhood of the surface of the Fermi Sea. Now, in the case of a non-ideal Fermi gas, the inter-particle forces cause virtual transition of particles. Thus extra holes and excited particles appear. But if the range of inter-particle force is not too short and the force itself is not too strong, these virtual holes and excited particles are still present only in the neighborhood of the Fermi maximum. If we confine ourselves to states of such type, we can simplify the commutation relations (2.13) – (2.15) in the following manner. Let us consider, for instance, the commutation relations $[\rho_1(k), \rho_1(k')]$ for which $k > 0$ and $-k < k' < k$.

We notice that the expression on the RHS of (2.8)

is a sum of operators each bringing one particle from the level $p+k'+\frac{k}{2}$ *to the level* $p-k'-\frac{k}{2}$.

Because the summation over p is extended only between k' *and* $\frac{k}{2}$, *the final levels* $p-k'-\frac{k}{2}$ lie in a limited interval between $-\frac{k}{2}$ *and* $-k'$.

Now let $k_{\max}$ denote the value of $|k|$ at the Fermi maximum. Then, if k and $|k'|$ are both sufficient ly small compared with $k_{\max}$, the levels $-\frac{k}{2}$ *and* $-k'$ both lie deep in the bottom of the Fermi sea where there are holes. In such a case the operator $a^{+}_{p-\frac{k}{2},s}\, a_{p+k'+\frac{k}{2},s}$ will give a vanishing result becuase the final level is occupied. Thus, for the commutator s $[\rho_1(k)\ , \rho_1(k')]$ are equivalent to zero [12].

We next consider $[\rho_1(k)\ , \rho_1(-k)] = \sum_{s}\sum_{-\frac{k}{2}\le p\le\frac{k}{2}} n_{p,s}$ (2.10) where $n_{p,s}$ is the occupation number of the level p. Since the level p, which lies between $-\frac{k}{2}$ *and* k, lies deep in the Fermi sea if k is small compared with $k_{\max}$, it is occupied by one particle. Then the sum $\sum n_{p,s}$ is simply equal to the number of levels between $-\frac{k}{2}$ *and* k, which is just k. So, we obtain $[\rho_1(k)\ , \rho_1(-k)]$ as equivalent to k (2.13).

The Hamiltonian (2.1) has been *solved approximately*. Only the excitation spectrum has been obtained. And it is worth noting that, some of these excitations are fluctuations [13], in the density operator $\rho_{(k)}$. Very similar results to the Tomonaga model are obtained by writing an equation of motion for the density operator and solving it approximately [14].

Taking the interaction potential as a constant, $V_k = V_0$ the energy spectrum is just altered by having the Fermi velocity increased as shown in (2.3). The constant V_o is assumed to be positive, since it describes interactions between electrons. The interactions increase the velocity of the acoustic plasmon. And with the interaction potential $V_k = \frac{2}{3}\left(\frac{e^2k_F^2}{k^2}\right)$, we have long-wavelength modes with a constant frequency, which is the plasma frequency (2.41).

In the electron gas, there are two different types of excitations. One is the plasma mode at long wavelength, and the other is the electron hole excitations at shorter wavelength. The latter are probably best described by the choice $V_k = V_o$ [8,15]

4.0 Conclusion

In this work, we have shown that the original Hamiltonian for the one-dimensional interacting electron gas that was not exactly solvable can be transformed into an exactly solvable Hamiltonian [8]. The physics of the excitations of the electron gas as approximate bosons [16], was used to solve the Hamiltonian exactly by changing the boson operators to a coordinate representation, and the eigenfrequency was obtained.

Intensive research on 1D electronic systems has raised fundamental issues as to the nature of quasi-particles and electron correlations in condensed matter. A variety of quantum phenomena have been found for quasi-1D compounds [17]. Tomonaga-Luttinger type electric conductivity has been observed for $BaVS_3$ [18], Peierls instability for $CuGeO_3$ [19], charge and spin density waves for $(TMTSF)_2AsF_6$ and TTF-TCNQ [20]. These diverse and interesting phenomena will continue to be of research interest to physicists.

Acknowledgement

I appreciate the useful contributions received from Dr. B. E. Iyorzor, Famous Akpojotor and other members of the Theoretical Research group in the Department of Physics, University of Benin.

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