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Some Maps and their Chaoticity

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Abstract

The work shows the determinant of the standard map and logistic map with their chaoticity. The equations of the maps were iterated at various different values for parameters k (stochascity for standard map) and r (stochascity for logistic map) at different values. It was noticed that the chaoticity of the standard depend on the parameter k and the initial values of X and θ in the equation. The logistic map iteration shown the result of the map is only dependent on the parameter r not on the initial values of X chosen for the equation.

Keywords: Chaos, Standard map, Logistics map, Stochascity.

1.0 Introduction

Chaos theory describes the behaviour of certain dynamical systems, that is, systems whose state evolves with time that they may exhibit dynamics that are highly sensitive to initial conditions. As a result of this sensitivity, which manifests itself as an exponential growth of perturbations in the initial conditions, the behaviour of chaotic systems observed to be random. The chaos theory method from Lorenz and Poincare is a technique that can be used for studying complex and dynamical systems to reveal pattern of order (non-chaos) out of seemingly chaotic behaviours [1]. These chaotic systems can be studied through nonlinear maps, such as standard map and logistic map.

The mapping, known as standard map, was first studied by chirikov. On the basis of the behaviour of this map chirikov proposed in 1979 a criterion determining the transition to global stochasticity in Hamiltonian systems [2]. The origin of the well known and widely used standard map lies in the field of particle physics. The problem examined by Fermi, as an analogue to a possible cosmic ray acceleration mechanism in which charged particles are accelerated by collision with moving magnetic field structures, is that of a ball bouncing between a fixed and an oscillating wall. For every impact of the ball on the wall, the phase of the oscillation is chosen at random, the ball will get accelerated. The question was now that, if the ball would be also accelerated, when the wall oscillation is a periodic function of time. This problem was investigated by Ulam who found that the particle motion appeared to be stochastic, but did not increase its energy [3].

Logistic map is a simple different equation, which is a discrete map defined over a unit interval [4]. It describes population dynamics, or rise and decline of population interacting with each other through predator-prey relationship. Its solutions exhibit regularities as well as chaotic behaviour [5]. In a fascinating and influential review article, Robert May (1976) emphasized that even a simple nonlinear map could have very complicated dynamics [6]. The study intends to discuss standard map and logistic map and their chaoticity in simple language for everyone to understand.

2.0 Deterministics Chaos

There are two requirements for a solution that exhibits deterministic chaos. The first requirement is that we are solving deterministic equation with specified initial and boundary conditions. The second is that solutions that have initial condition that are infinitesimally close diverge exponentially as they evolve. The existence of deterministic randomness is the ultimate reason for the apparent chaotic behavior of the solutions of certain deterministic systems or process, and resulted in the name deterministic chaos [7].

The concept of deterministic chaos bridges the gap between stable deterministic solutions to equations and deterministic solution that are unstable to infinitesimal disturbances [8].

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2.1 The Standard Map

The standard map is an area-preserving chaotic map for two dynamical variables, angular momentum and angular position. Patidar and Sud [3] said the exact Ulam mapping for the motion of the ball bouncing between a fixed and an oscillating wall, where the velocity is defined by a saw tooth function, is given by a set of four exact difference equations. Under the area-preserving condition and if we allow the wall to add momentum to the ball according to its velocity (without a change in the position of the wall), this set of four difference equation be converted into a set of two difference equations:

$$U_{n+1} = |U_n + sin\Psi_n| \quad (1)$$

$$\frac{\Psi_{n+1}}{U_{n+1}} = \Psi_n + 2\pi M \quad (2)$$

Where u_n and Ψ_n respectively, are directly related to the ball velocity and phase just before the nth impact. The standard mapping is obtained by the linearization of the above set of difference equations in action space. The set of difference equations thus obtained are

$$X_{n+1} = X_n + KsinY_n \quad mod\ 2\pi \quad (3)$$

$$Y_{n+1} = Y_n + X_{n+1} \quad mod\ 2\pi \quad (4)$$

We used the difference equations below to study standard map;

$$\boldsymbol{X_{n+1} = X_n - Ksin2\theta_n \quad mod\ 1} \quad (5)$$

$$\boldsymbol{\theta_{n+1} = \theta_n + X_n \quad mod\ 1} \quad (6)$$

It can also be referred to as mapping phase space into a phase space. The $\boldsymbol{X_n}$ and $\boldsymbol{\theta_n}$ are the initial values while $\boldsymbol{\theta_{n+1}}$ and $\boldsymbol{X_{n+1}}$ are the new values after one map iteration, and K is a dimensionless parameter that influences the degree of chaos. Different values of $\boldsymbol{K}$ were chosen with different initial values of $\boldsymbol{X}$ and $\boldsymbol{\theta}$ to test the resulting map of each $\boldsymbol{K}$. To get the accurate values for subsequent $\boldsymbol{X}$ and $\boldsymbol{\theta}$, modulo of $\mathbf{1}$ was taken which really defined the standard map.

2.2 The Logistic Map

The logistic map is a very simple mathematical model often used to describe the growth of biological populations [9]. Logistic map is a difference equation of the form,

$$\boldsymbol{X_{n+1} = f(X_n)} \quad (1)$$

By assigning value $\boldsymbol{X_1}$ for $\boldsymbol{X}$, the above equation mean that $\boldsymbol{X_2 = f(X_1)}$, $\boldsymbol{X_3 = f(X_2)}$, so on. The $\boldsymbol{f(X_n)}$ in the above equation is $\boldsymbol{r(1 - X_n)X_n}$so the difference equation is now

$$\boldsymbol{X_{n+1} = r(1 - X_n)X_n} \quad (2)$$

This equation is associated to the logistic continous equation originally introduced by P. F. Verhulst as a model of population growth [10].

To study the dynamics of the logistic map we must look at its long-term behaviour (its attractors). That is, we want to iterate the map over and over again and watch what happens to $\boldsymbol{X}$. We must do this for various values of r with different initial values of $\boldsymbol{X}$, and see how the pattern change as r is varied with different initial values of $\boldsymbol{X}$. We choose $\boldsymbol{r}$ to be 1.5, 2, 3, 3.7, 4 and 4.1 with different initial values at 0.2887 and 0.7071 of $\boldsymbol{X}$. The various behaviours were observed. The parameter $\boldsymbol{r}$ which is the growth rate of the equation is the key factor that determines the degree of chaos of this dynamical difference equation.

Qualitative results 2-dimensional phase space for standard map

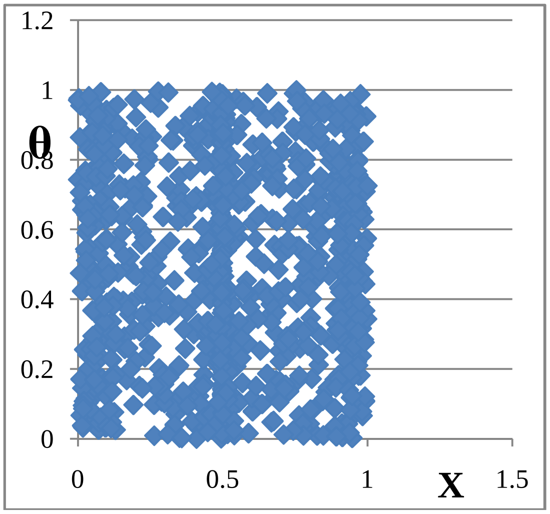

Figure 1 **K=0.6, X_O=0.3, Θ_O=0.5**
Some points were concentrated around X=0.5, the area of more concentration is around X=0.95, when ϴ is between 0.6 and 0.8. These were chaos effect more tense.

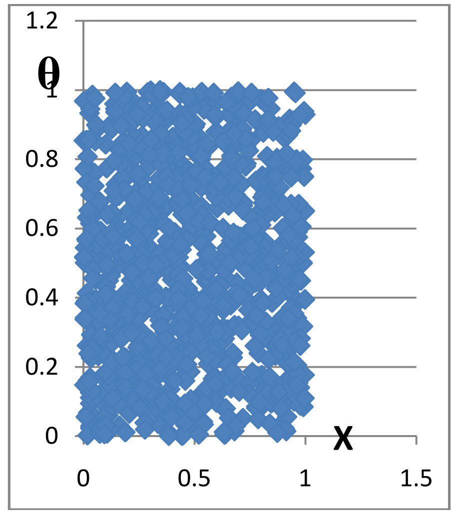

Figure 2 **K=4.3, X_O=0.3, Θ_O=0.5.**
The points were distributed over the phase space.

Qualitative results: 2-dimensional phase space for logistic map

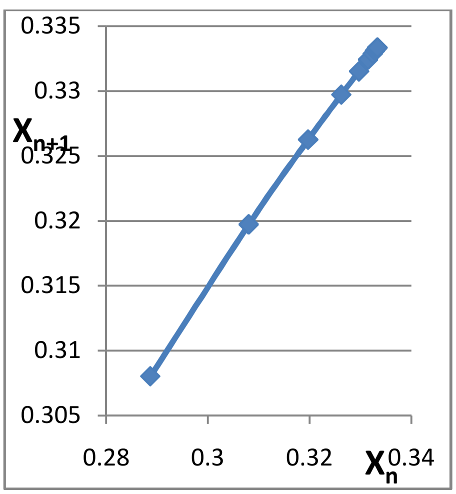

Figure 3 **r= 1.5, X_O=0.2887.** Attractor is 0.333.
The map is stable on fixed point 0.333, after little iteration.

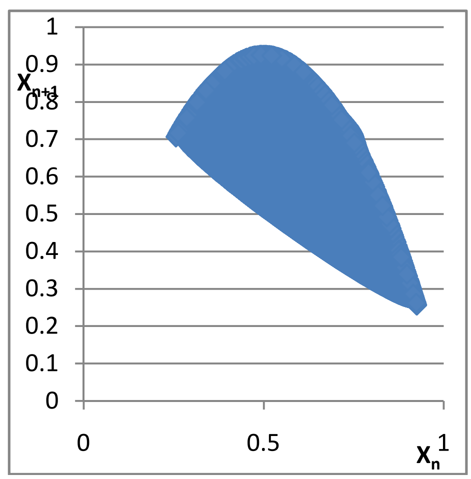

Figure 4 **r=3.7, X_O=0.2887.** Attractor is 0.73
The map is highly oscillating, exhibit chaotic behaviour

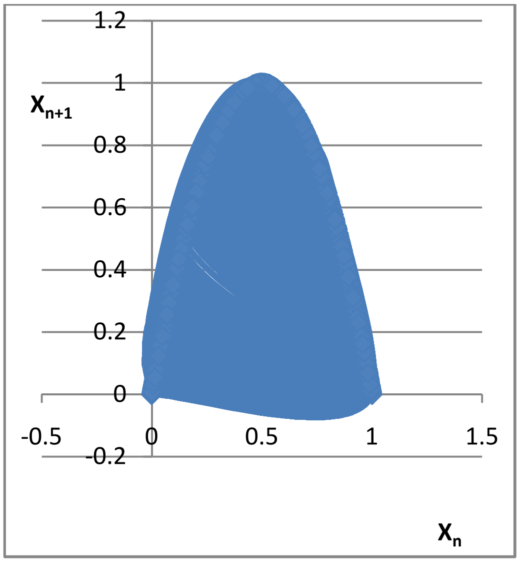

Figure 5 r=4, X_O=0.2887. Attractor is 0.75
The map is highly chaotic, the oscillation so much and dense.

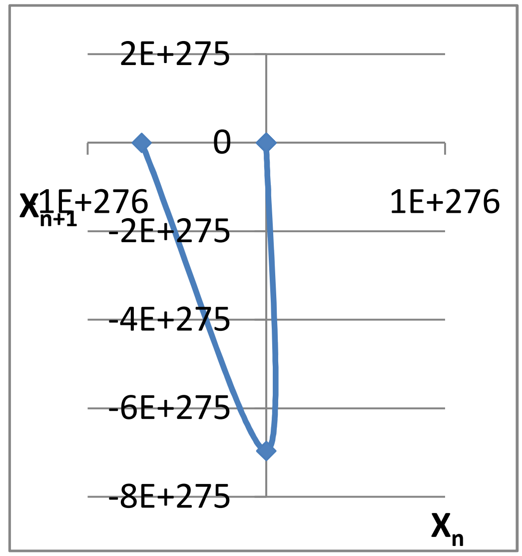

Figure 6 r=4.1, X_O=0.7071. Attractor is 0.756.
The map diverged totally from the fixed point; the Values leave the interval [0, 1].

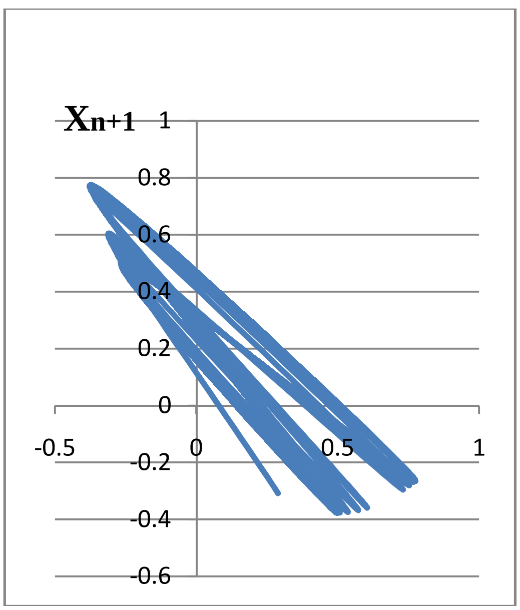

Figure 7, r = -1.5, X_0 = 0.2887, attractor is 1.667
The map iteration did not get to the fixed point or the attractor

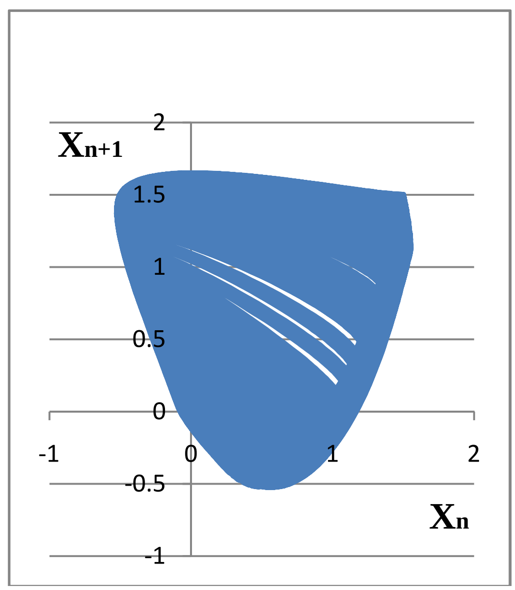

Figure 8, r = -2, X_0 = 0.7071. Attractor is 1.5
The map is highly chaotic, the oscillation so much and dense.

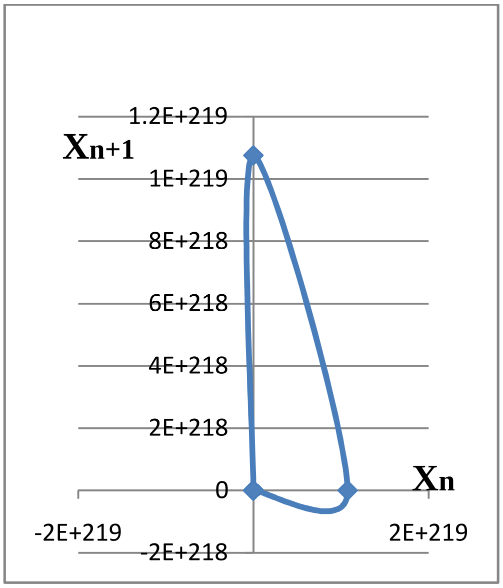

Figure 9, r = -3.0, X_0 = 0.2887, attractor is 1.333
The map diverge totally from the fixed point, the values
values leave the interval [0, 1]

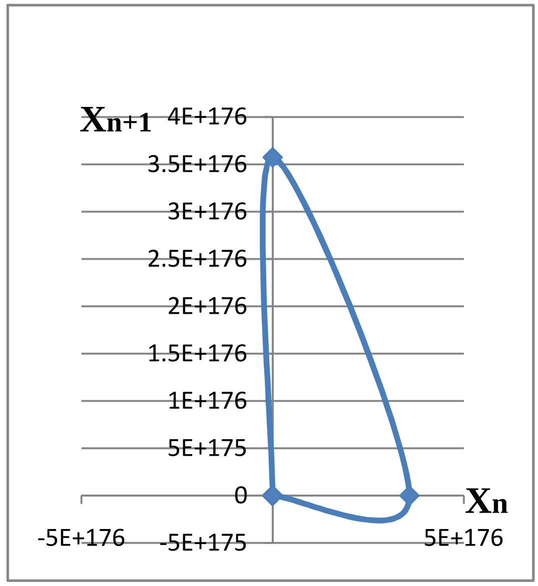

Figure 10, r = -4.1, X_0 = 0.7071, the attractor is 1.244
The map diverge totally from the fixed point, the

leave the interval [0, 1]

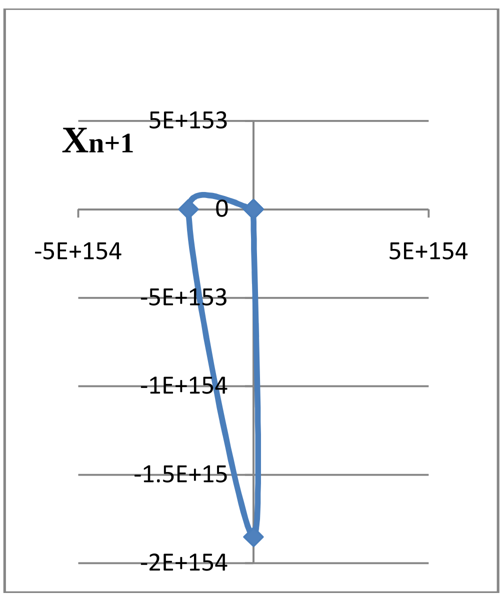

Figure 11, r = 6.2, X_0 = 0.2887, the attractor is 0.839
The map diverge totally from the fixed point, the values leave the interval [0, 1]

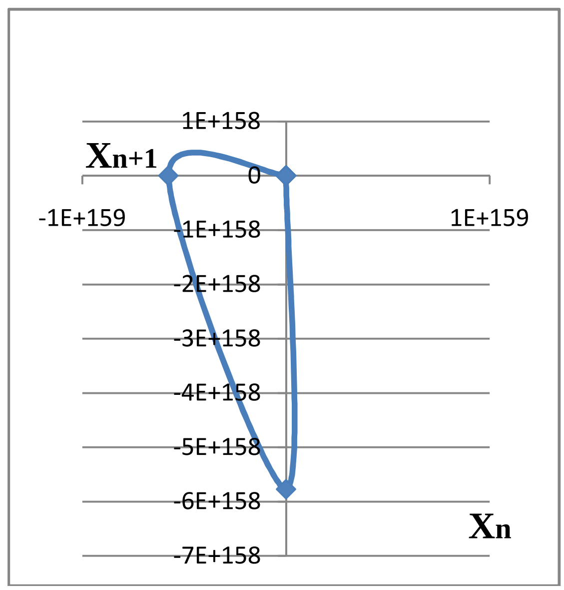

Figure 12, r = 10.7, X_0 = 0.7071, the attractor is 0.907
The map diverge totally from the fixed point, the values leave the interval [0, 1]

Discussion

The results of the iteration of standard map were graphically explained using figure 1 and figure 2. In Figure 1, $K = 0.6, X_o = 0.3, \theta_0 = 0.5$, some points were much concentrated around $X = 0.5$. This region is more chaotic than others in the phase space. While in Figure 2, $K = 4.3,\ X_0 = 0.3,\ \theta_0 = 0.5$, some points were concentrated around X=0.5, few points were concentrated around X=0.95 when θ is between 0.6 and 0.8. It shows that the degree of their chaoticity depends on the parameter k.

The results of logistic map were explained graphically using Figure 3 to Figure 7. In Figure 3, $r = 1.5$ the map is stable at the fixed point; this could be referred to as the attractor of the map. As the value of r increases, chaotic behavoiur is observed indicating that the values were oscillating in much time when $r = 3.7$ as shown in fig. 4.

When $r = 4$ a more chaotic system is observed, and is oscillating at different values, i.e. it is not stable on the fixed point as seen in Figure 5. Figure 6 shows the result of $r = 4.1$, the values diverged from the attractor, leaving the interval [0,1]. The results of figures 1 to 6 agrees with previous research work carried out on standard and logistics maps.

As a step further, we considered cases where the value of r is negative (Fig. 7 to 10). This has not been treated in previous literatures to the best of our knowledge. When r = -1.5 for all the initial values used, the system is beginning to be periodic and chaotic in behaviour. When r = -2.0, the graphs for all the initial values used are the same, the periodicity is highly dense and so chaotic in behaviour. When r = -3.0 and -4.1 with the initial values used, the system is within the interval of 0 and 1 and diverge to positive infinite. It shows that r beyond -2.0 the values obtained will tend to infinite.

We also went further to consider cases of higher positive values of r between 5.0 and 10.7. When r = 6.2 and 10.7 the values leave the interval of 0 and 1 and diverge to negative infinite as shown in figures 11 and 12. This shows that logistic map is dependent on r and not on the initial values of X used.

Conclusion

It was noticed in this work that, in standard map and logistic map, the degree of chaos depends on the dimensionless parameters K and r respectively. In the standard map, initial values of X and θ are important, but when $K = 3.4$ and $K = 4.3$, the degree of chaotic strongly depend on K not on initial value of X and θ. While in logistic map the degree of chaos depends on the value of r. At different values of X with the value of the same value of r, we got the same results.

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