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A panacea to the asymptotic effect of examination malpractice in a malpractice-stamp-out scenario

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Abstract

This study provides a solution to a recent result by Ekhosuehi and Osagiede [1] on the asymptotic effect of examination malpractice. We prove that in a dense population of candidates seeking admission in an environment saddled with examination malpractice, educational institutions can maintain the enrolment structure at a certain level $\mathbf{n}^*$ if a specific quota is fixed by the Ministry of Education or its regulating agency for new entrants into the system.

Keywords: absorbing Markov chain; examination malpractice; multi-echelon educational system; terminal enrolment structure.

1.0 Introduction

Ekhosuehi and Osagiede [1] considered the effect of examination malpractice on a malpractice-free educational institution with a set of possible states $S = \{s_1, s_2, \ldots, s_k\}$, wherein:

i. $s_i < s_j$, for $i < j$, $i, j = 1, 2, \ldots, k$;

ii. $\{s_i\}$, $i = 1, 2, \ldots, k$, is a singleton set;

iii. $\{s_i\} \cap \{s_j\} = \phi$, $i \neq j$, where ϕ is a null set;

iv. $\#\left(\bigcup_{i=1}^{k}\{s_i\}\right) = k$, $k < \infty$; and

v. for an index collection of random variables $\{X_t\}$, where the index t runs through a given set T,

$$\mathrm{Prob}\{\mathrm{X}_{t+1} = s_j \mid \mathrm{X}_0 = s_1, \ldots, \mathrm{X}_t = s_i\} = \mathrm{Prob}\{\mathrm{X}_{t+1} = s_j \mid \mathrm{X}_t = s_i\},\ t = 0, 1, 2, \ldots,$$

and $\mathrm{Prob}\{\mathrm{X}_{t+1} = s_j \mid \mathrm{X}_t = s_i\} = p_{ij}$.

The educational institution was assumed to operate in an environment where examination malpractice is prevalent. The major accomplishment of Ekhosuehi and Osagiede [1] is formalized in the following theorem:

Theorem 1: *Given an environment saddled with examination malpractice and a Young/Almond-type educational system, where strategies to curb the malpractice rate are generally not effective, then, against every possible growth rate g, the students' stocks in each level of the educational institution where examination malpractice is prohibited will degenerate to zero in the long-run..*

Theorem 1 points to a doom for any educational institution that thrives towards stamping-out examination malpractice in an environment saddled with examination malpractice. To avert the consequences of Theorem 1, stakeholders in the educational sector need to brainstorm so as to find solutions to the problem. As a panacea to the problem, we propose the quota admission system where a fixed number of new entrants are allocated to each school on certain criteria. By doing this, the enrolment structure of schools where examination malpractice is strictly prohibited will not degenerate to zero. We shall prove this assertion using absorbing Markov chain. The use of absorbing Markov chain is prominent in analyzing transitions in the educational system. The works [2 – 6] are just a few references where absorbing Markov chain has been used in modeling the educational process.

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2.0 The theoretical preliminaries

Let R be a state-transition relation on the set $\Re = S \cup \{0\}$, such that $R \subset \Re \times \Re$. Let p_{ij} be the transition probability from state i to state j. Then, we can form a rectangular array, $\mathbf{A} = (a_{ij})$, whose rows and columns are labelled by the elements of $\Re$, where a_{ij} is given as

$$a_{ij} = \begin{cases} p_{ij} & \text{if } \exists \text{ a relation from } i \text{ to } j,\ i, j \in \Re \\ \\ 0 & \text{otherwise} \end{cases}.$$

From the foregoing, the arrangement of the transition probabilities of the relation R on $\Re$ is given as:

$$\mathbf{A} = \begin{array}{c} \\ \text{Nonabsorbing} \\ \\ \text{Absorbing} \end{array} \begin{array}{c} \qquad\text{Nonabsorbing} \qquad\qquad\qquad \text{Absorbing} \\ \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1k} & \vdots & p_{10} \\ p_{21} & p_{22} & \cdots & p_{2k} & \vdots & p_{20} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ p_{k1} & p_{k2} & \cdots & p_{kk} & \vdots & p_{k0} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \ldots & 0 & \vdots & 1 \end{bmatrix} \end{array}.$$

The transition matrix $\mathbf{A}$ is called the absorbing Markov chain since it contains an absorbing state. Let $\mathbf{w} = \begin{pmatrix} p_{10} & p_{20} & \ldots & p_{k0} \end{pmatrix}$ be the wastage vector and $\mathbf{P} = (p_{ij})$ be the transition probability matrix of the educational system. Then matrix $\mathbf{A}$ can be represented as $\mathbf{A} = \begin{bmatrix} \mathbf{P} & \vdots & \mathbf{w}' \\ \cdots & \cdots & \cdots \\ \mathbf{0} & \vdots & 1 \end{bmatrix}$, where $\mathbf{0}$ is a $1 \times k$ vector of zeros and the prime denotes transposition. By the matrix multiplication of a partitioned matrix, we have in general that $\mathbf{A}^t = \begin{bmatrix} \mathbf{P}^t & \vdots & \mathbf{w}_t^* \\ \cdots & \cdots & \cdots \\ \mathbf{0} & \vdots & 1 \end{bmatrix}$, where $\mathbf{w}_t^* = \mathbf{w}' + \mathbf{P}\mathbf{w}' + \mathbf{P}^2\mathbf{w}' + \cdots + \mathbf{P}^{t-1}\mathbf{w}'$. If $t \to \infty$, $\lim_{t\to\infty} \mathbf{P}^t = \mathbf{0}$ since $\|\mathbf{P}\| < 1$ as $\mathbf{P}$ is sub-stochastic and $\lim_{t\to\infty} \mathbf{w}_t^* = \mathbf{w}' + \mathbf{P}\mathbf{w}' + \mathbf{P}^2\mathbf{w}' + \cdots = (\mathbf{I} - \mathbf{P})^{-1}\mathbf{w}'$. The matrix $(\mathbf{I} - \mathbf{P})^{-1}$ is called the fundamental matrix of the absorbing Markov chain and its (i, j)th entry is the expected number of sessions a student in level i will stay before leaving to level j. The fundamental matrix of the absorbing Markov chain had earlier been employed by Uche [5] to estimate the cost of education. To estimate the transition probabilities, it is often assumed that flows within the system under consideration is a random variable with a multinomial (or Dirichlet) distribution [7 – 8] of the form

$$P(n_{i0}(t), n_{i1}(t), \ldots, n_{ik}(t)) = \frac{\Gamma\left(\sum_{j=0}^{k} n_{ij}(t) + 1\right)}{\prod_{j=0}^{k} \Gamma(n_{ij}(t) + 1)} \prod_{j=0}^{k} (p_{ij}(t))^{n_{ij}(t)}.$$

The function Γ is the gamma function defined for x say, as $\Gamma(x) = \int_0^\infty t^{x-1} \exp(-t)\,dt$, and for $x \in \mathbf{Z}^+$, $\Gamma(x+1) = x!$. The Dirichlet distributions are natural choices to analyse data described by frequencies or proportions [9] and they are conjugate prior of multinomial distributions in Bayesian statistics. By conjugate prior, we mean that, in Bayesian probability theory, the Dirichlet distribution is the prior of the likelihood function, since the multinomial (posterior) probability and the Dirichlet distribution belong to the same exponential family. The transition probabilities are then estimated using the method of maximum likelihood. In order to realise the assumption that the transition probabilities of the Markov chain are constant over time, the estimates are pooled to obtain the minimum variance estimators.

Bartholomew et al. [7] had earlier identified the Markov family of models as models suitable for 'push' flows wherein the models are used to investigate how the grade sizes of a system would change under the operation of constant average flow rates. Push flows are flows in which the impetus for a move lies at its starting-point. The educational system is one of such 'push' flow systems as the move to a higher grade is occasioned by accumulating a certain minimum credits requirement at the lower grade. In particular, the discrete time Markov models have practical relevance for the educational system since data on students' transitions are often recorded at the end of the academic year. Markov models with the assumption of constant flow rates provide a useful platform for describing the future trajectory in the educational system. This is because in such Markov models, the steady state distribution can be obtained. The practical interests of the steady state distribution are that it tells the researcher the direction in which the system is moving and it gives the picture of the relationship between the stocks and flows in order for the system to remain stable over time.

The classical Markov model for the educational system was developed by Gani [2] for the Australian university system. Gani [2] considered a cohort N_{t-j} (i.e. new enrolments at times $t-j$, $j=i-1, i, i+1, \ldots$) of students enrolling for the first time at the beginning of the year $t-j$, where i applies to the states in the degree programme under consideration. The model is presented as: $S_{1t} = N_t + p_{11}S_{1,t-1}$,

$S_{it} = p_{i-1,i}S_{i-1,t-1} + p_{ii}S_{i,t-1} \quad (i = 2,3,\ldots,7)$, or, in matrix form as: $\mathbf{S}_t = \mathbf{P}\mathbf{S}_{t-1} + \mathbf{E}\,N_t$, where $\mathbf{E}$ is the column vector with the first element 1 and other elements equal to zero, $\mathbf{S}_t$ is a column vector with element S_{it} which is the number of students in the i th year at the start of year t; $\mathbf{P} = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1k} \\ p_{21} & p_{22} & \cdots & p_{2k} \\ \cdot & \cdot & \cdot & \cdot \\ p_{k1} & p_{k2} & \cdots & p_{kk} \end{bmatrix}^T$, p_{ij} is the proportion of students enrolled in i th year at time t re-enrolling in j th year at time $t+1$ ($p_{ij} > 0$ for $j = i, i+1$; and zero otherwise) and T denotes transposition. Since the work of Gani [2], many similar models have been discussed in literature [3 – 6]. These works rely on the fundamental of an absorbing Markov chain as the transition matrix $\mathbf{P} = (p_{ij})$ is sub-stochastic. In the absorbing models, the recruitment probabilities, transition probabilities, initial structure and recruits are assumed given or can be estimated.

3.0 The quota admission system as a panacea to a degenerating enrolment structure

Ekhosuehi and Osagiede [1] examined the effect of examination malpractice on the structure of the educational system by modifying the basic absorbing Markov chain model as

$$\bar{\mathbf{n}}(t+1) = \mathbf{n}(t)\mathbf{P} + \rho R(t+1)\mathbf{P_0} \quad \text{for} \quad t \in \{0\} \cup \mathbf{Z}^+,\ \bar{\mathbf{n}}(t+1) \in \mathbf{R}^k_{\geq 0},$$

where $\mathbf{n}(t) = (n_1(t),\ n_2(t),\ \cdots,\ n_k(t))$ is a point in the $k-$dimensional Euclidean space with $n_i(t)$ being the number of individuals in state $i \in S$ in period t, $\bar{\mathbf{n}}(t+1)$ is the expected manpower structure in period $t+1$, $\mathbf{R}^k_{\geq 0}$ is the set of non-zero real numbers in the $k-$dimensional Euclidean space, $\mathbf{Z}^+$ is the set of positive integers, ρ is a parameter on the confidence level on the entrance examination through which a successful candidate is admitted into the educational system, $\mathbf{P}_0$ is the recruitment vector, and $\mathbf{P}$ is a sub-stochastic transition matrix given as

$$\mathbf{P} = \left\{(p_{ij}) : \sum_{j=1}^{k} p_{ij} \leq 1,\ p_{ij} \geq 0,\ i, j \in S\right\}.$$

We now formalise the main contribution of this paper in the following theorem.

Theorem 2: *Given a dense population,* $\Omega(t+1)$*, of candidates seeking admission in an environment saddled with examination malpractice, educational institutions can maintain the enrolment structure at a certain level* $\mathbf{n}^*$*, if a certain quota is fixed by the Ministry of Education or its regulating agency for new entrants into the system.*

Proof

By a standard argument [1],

$$\bar{\mathbf{n}}(t+1) = \mathbf{n}(t)\mathbf{P} + \rho R(t+1)\mathbf{P_0} \quad \text{for } t \in \{0\} \cup \mathbf{Z}^+.$$

If a certain quota is fixed by the Ministry of Education or its regulating agency for new entrants into the system with effect from the period t^*, then

$$R(t+1) = R < \Omega(t+1) \quad \forall\, t \geq t^*.$$

Thus $\bar{\mathbf{n}}(t+1) = \mathbf{n}(t)\mathbf{P} + \rho R\mathbf{P_0}$. Consider the three successive iterates $\cdots, \bar{\mathbf{n}}(t), \bar{\mathbf{n}}(t+1), \bar{\mathbf{n}}(t+2), \cdots$, where $\bar{\mathbf{n}}(t+1) = \mathbf{n}(t)\mathbf{P} + \rho R\mathbf{P_0}$ and $\bar{\mathbf{n}}(t+2) = \mathbf{n}(t+1)\mathbf{P} + \rho R\mathbf{P_0}$.

Then $\bar{\mathbf{n}}(t+2) - \bar{\mathbf{n}}(t+1) = (\mathbf{n}(t+1) - \mathbf{n}(t))\mathbf{P}$. Taking the norm, we have

$$\left\|\left(\bar{\mathbf{n}}(t+2) - \bar{\mathbf{n}}(t+1)\right)\right\| = \|(\mathbf{n}(t+1) - \mathbf{n}(t))\|\|\mathbf{P}\|.$$

The iteration shrinks the change $\left\|\left(\bar{\mathbf{n}}(t+2) - \bar{\mathbf{n}}(t+1)\right)\right\|$ to zero because the matrix $\mathbf{P}$ is substochastic. If $\mathbf{n}^*$ is the solution such that $\mathbf{n}^* = \mathbf{n}^*\mathbf{P} + \rho R\mathbf{P_0}$, then replacing $\bar{\mathbf{n}}(t+1)$ and $\mathbf{n}(t)$ by $\mathbf{n}^*$, we obtain

$$\left\|\left(\bar{\mathbf{n}}(t+2) - \mathbf{n}^*\right)\right\| \leq \|(\mathbf{n}(t+1) - \mathbf{n}^*)\|\|\mathbf{P}\| < \|(\mathbf{n}(t+1) - \mathbf{n}^*)\|$$

which implies that the iterates are converging to $\mathbf{n}^*$. This completes the proof.

We shall, hereafter, refer to $\mathbf{n}^*$ as the terminal enrolment structure. The terminal enrolment structure $\mathbf{n}^*$ can be computed for $R(t+1) = R < \Omega(t+1) \quad \forall\, t \geq t^*$ from the equation $\mathbf{n}^* = \mathbf{n}^*\mathbf{P} + \rho R\mathbf{P_0}$. Thus, we have $\mathbf{n}^* = \rho R\mathbf{P_0}(\mathbf{I} - \mathbf{P})^{-1}$, provided $\det(\mathbf{I} - \mathbf{P}) \neq 0$. Since the solution $\mathbf{n}^* = \rho R\mathbf{P_0}(\mathbf{I} - \mathbf{P})^{-1}$ may contain non-integral entries, we modify the result as: $\mathbf{n} = \text{ceiling}\,[\rho R\mathbf{P_0}(\mathbf{I} - \mathbf{P})^{-1}]$, where $\text{ceiling}[\mathbf{x}]$ is a vector in which its entries are the smallest integer greater than or equal to the corresponding elements of $\mathbf{x}$.

We illustrate the assertion in Theorem 2 using the example problem in [1], where ρ is obtained as $\rho = 0.25$, $\mathbf{P}_0 = [0.8624 \quad 0.1376 \quad 0 \quad 0 \quad 0 \quad 0]$ and

$$\mathbf{P} = \begin{bmatrix} 0.0486 & 0.9060 & 0 & 0 & 0 & 0 \\ 0 & 0.0696 & 0.7890 & 0 & 0 & 0 \\ 0 & 0 & 0.1187 & 0.8249 & 0 & 0 \\ 0 & 0 & 0 & 0.0948 & 0.8436 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.9173 \\ 0 & 0 & 0 & 0 & 0 & 0.2947 \end{bmatrix}.$$

Taking $R = 150$ and $\mathbf{n}(0) = (110 \quad 112 \quad 53 \quad 56 \quad 30 \quad 43)$ as the base enrolment structure, we demonstrate the utility of our assertion in Theorem 2 with Matlab R2007b (see appendix) and therefore, obtain a graph of the ten-year enrolment projected structure for the quota admission process as depicted in Fig. 1.

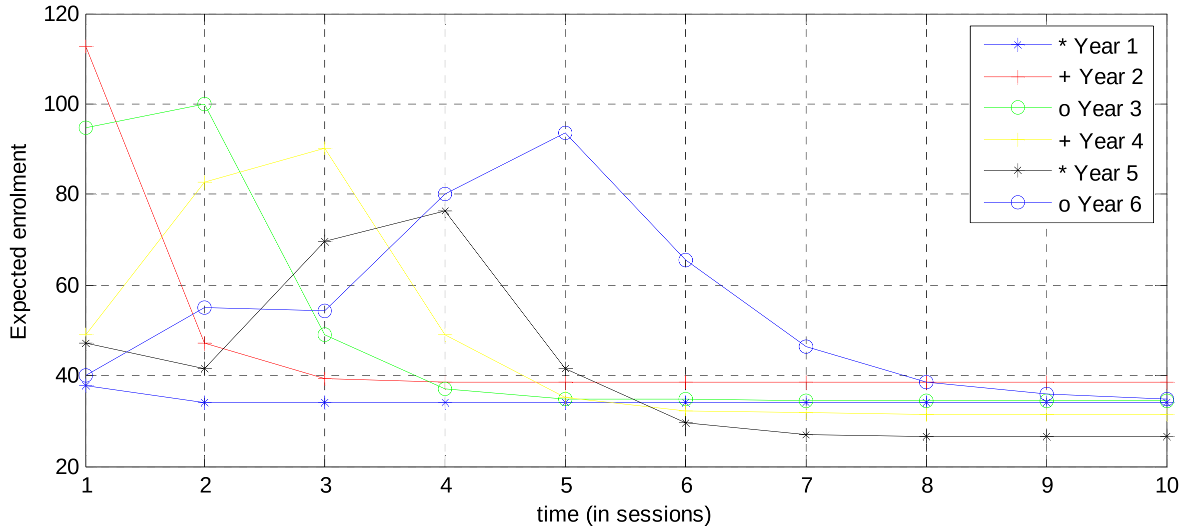

Fig. 1: A ten-year projected enrolment structure with $R = 150$.

Fig. 1 shows how the actual grade sizes vary over the ten-year projection period when a quota $R = 150$ is fixed for new entrants. From Fig. 1, enrolment in Year 1 tends to a constant figure after the first period of projection, while enrolment in Year 2 is very high at the first projection period and then decreases suddenly in the second period and thereafter settles at a constant figure after the third period. The constant figure in each of Year 1 and Year 2 is the terminal enrolment for that year. Enrolment in Year 2 is higher than that of Year 1 throughout the projection period because new entrants are admitted into Year 2 to join students promoted from Year 1. The enrolment trajectory of Year 2 has a ripple effect in Year 3 – 6 because the transition pattern is such that $p_{ij} \geq 0$ for $1 \leq i \leq 6$, $j = i, i+1$; and zero otherwise as indicated by the transition matrix $\mathbf{P}$. The terminal enrolment for Year 3 – 6 is however lower than that of Year 2. This is due to wastage in the system. In all, the entire projections for $t \leq 5$ fluctuate tremendously; and afterwards, there is a steady decline in the enrolment structure until it reaches its terminal enrolment. This result agrees reasonably well with the assertion in Theorem 2. Thus we conclude that when the admission stock is fixed, the structure of the system will not degenerate to zero, but tends to a certain non-zero structure. In this scenario, institutions where examination malpractice is strictly prohibited can still remain viable as they are hopeful of their survival in the competitive fringe. We therefore compute the terminal enrolment structure as $\mathbf{n} = \begin{pmatrix} 34 & 39 & 35 & 32 & 27 & 35 \end{pmatrix}$.

4.0 Conclusion

In Theorem 2 and the application, our emphasis is the necessity to set a quota for new entrants into the educational system in the wave of examination malpractice. By so doing, a ray of hope holds for educational institutions where examination malpractice is strictly prohibited. To implement this assertion in practice, the Ministry of Education or its regulating agency, should workout modalities for a central entrance examination into educational institutions, where successful candidates are allotted to the institutions in line with the laid down quota.

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References

[1] Ekhosuehi, V. U. and Osagiede, A. A. (2011). 'On the asymptotic effect of examination malpractice on the structure of a Markovian multi-echelon educational system', *International Journal of Mathematics in Operational Research*, Vol. 3, pp. 619-635.

[2] Gani, J. (1963). 'Formulae for projecting enrolments and degrees awarded in universities', *Journal of the Royal Statistical Society. Series A (General)*, Vol. 126, No. 3, pp: 400 - 409.

[3] Nicholls, M. G. (1983). 'A Markovian evaluation of a tertiary education faculty', *Higher Education* 12, pp: 721-730.
[4] Nicholls, M. G. (2009). 'The use of Markov models as an aid to the evaluation, planning and benchmarking of doctoral programs', *Journal of the Operational Research Society*, Vol. 60, pp: 1183-1190.
[5] Uche, P. I. (1978). 'On stochastic models for educational planning', *International Journal of Mathematical Education in Science and Technology*, Vol. 9, No. 3, pp: 333-342.
[6] Uche, P. I. (2000). The Markovian model of the educational process. In: A. O. Animalu, S. O. Iyahen and H. O. Tejumola (Editors), *Contribution to the Development of Mathematics in Nigeria*. National Mathematical Centre, Abuja. Pp: 235-245.
[7] Anderson, T. W. and Goodman, L. A. (1957). 'Statistical inference about Markov chains', *Annals of Mathematical Statistics*, Vol. 28, No. 1, pp: 89-110.
[8] Bartholomew, D. J., Forbes, A. F., and McClean, S. I. (1991). *Statistical techniques for manpower planning* (2^{nd} ed.). John Willey & Sons: Chichester.
[9] Wicker, N., Muller, J., Kalathur, R. K. R., and Poch, O. (2008). 'A maximum likelihood approximation method for Dirichlet's parameter estimation', *Computational Statistics & Data Analysis*, Vol. 52, issue 3, pp: 1315-1322.

Appendix: The Matlab codes

```
% Model parameters and analysis.
P =[0.0486  0.9060      0         0         0         0;
    0    0.0696    0.7890       0         0         0;
    0         0    0.1187    0.8249       0         0;
    0         0         0    0.0948    0.8436       0;
    0         0         0         0         0    0.9173;
    0         0         0         0         0    0.2947];

P0 =[0.8624    0.1376      0       0       0       0];
n0 =[110   112    53    56    30    43];
rho =0.2500;
Rn=150;

% The quota system enrolment projection.
f1=n0*P+rho*Rn*P0, f2=f1*P+rho*Rn*P0, f3=f2*P+rho*Rn*P0, f4=f3*P+rho*Rn*P0,
f5=f4*P+rho*Rn*P0, f6=f5*P+rho*Rn*P0, f7=f6*P+rho*Rn*P0, f8=f7*P+rho*Rn*P0,
f9=f8*P+rho*Rn*P0, f10=f9*P+rho*Rn*P0,
n=rho*Rn*P0*inv(eye(6)-P),

L1=[f1(1,1) f2(1,1) f3(1,1) f4(1,1) f5(1,1) f6(1,1) f7(1,1) f8(1,1) f9(1,1)
f10(1,1)];
L2=[f1(1,2) f2(1,2) f3(1,2) f4(1,2) f5(1,2) f6(1,2) f7(1,2) f8(1,2) f9(1,2)
f10(1,2)];
L3=[f1(1,3) f2(1,3) f3(1,3) f4(1,3) f5(1,3) f6(1,3) f7(1,3) f8(1,3) f9(1,3)
f10(1,3)];
L4=[f1(1,4) f2(1,4) f3(1,4) f4(1,4) f5(1,4) f6(1,4) f7(1,4) f8(1,4) f9(1,4)
f10(1,4)];
L5=[f1(1,5) f2(1,5) f3(1,5) f4(1,5) f5(1,5) f6(1,5) f7(1,5) f8(1,5) f9(1,5)
f10(1,5)];
L6=[f1(1,6) f2(1,6) f3(1,6) f4(1,6) f5(1,6) f6(1,6) f7(1,6) f8(1,6) f9(1,6)
f10(1,6)];
t1=1:10;
plot(t1,L1,'b*-',t1,L2,'r+-',t1,L3,'go-',t1,L4,'y+-',t1,L5,'k*-',t1,L6,'bo-')
xlabel('time (in sessions)')
ylabel('Expected enrolment')
legend('* Year 1', '+ Year 2', 'o Year 3', '+ Year 4', '* Year 5', 'o Year 6')
```