Journal of the Nigerian Association of Mathematical Physics
Volume **20 (March, 2012), pp 185 – 192**

Augmented Lagrangian Method For Discretized Optimal Control Problems

Olotu, O and Akeremale, O.C.

The Federal University of Technology, Mathematical Sciences Department, P.M.B.704, Akure, Ondo State, Nigeria

Abstract

In this paper, we are concerned with one-dimensional time invariant optimal control problem, whose objective function is quadratic and the dynamical system is a differential equation with initial condition .Since most real life problems are non-linear and their analytical solutions are not readily available, we resolve to approximate solutions. Our aim is to develop a numerical scheme to solve one dimensional optimal control problem. A discretization of the performance index using trapezoidal rule and the state equation using crank-Nicholson is adopted. By parameter optimization, this results into a sparse non-linear programming problem. With the aid of Augmented Lagrangian method, a quadratic function with a control operator (penalized matrix) amenable to conjugate gradient method is generated. Numerical experiments verify the efficiency of the proposed technique which compares much more favourably to the existing scheme.

Keywords: Trapezoidal rule, Euler's method, Augmented Lagrangian method, and Conjugate gradient method.

1.0 Introduction

Optimal control theory is an extension of calculus of variation. It is a mathematical optimization method for deriving control policies [1]. Optimal control deals with the problem of finding a control law for a given system such that certain optimality criterion is achieved. In the problem of optimal control, the trajectory is determined, which satisfies simultaneous equations of motions, boundary conditions, inequality constraints, equality constraints, where the performance index (cost functional) must be minimized or maximized. There are many procedures for solving optimal control problems such as calculus of variations, minimum principle, matrix exponential, and Hamilton-Jacobi equations. However, these are considered as indirect procedures, since the necessary and sufficient conditions must be derived and result expressible in differential-algebraic equation (DAEs). This paper focuses on the direct procedure in which the optimal control problems will be converted to parameter optimization problems. In section 2, the statement of the problem is described along with a technique developed by [2] called exterior penalty method. We are proposing a similar technique in section 3 called Augmented Lagrangian method. We believe that by using augmented Lagrangian, the problem of ill-conditioning attached to the perturbed matrix will be reduced and as such, a better result with lesser iterations will be obtained. The Augmented Lagrangian algorithm is shown in section 4. Finally, examples are illustrated in section 5 to show the efficiency of the new scheme compared to exterior penalty method.

2.0 General Formulation of The Problem

The statement of the problem is to find an optimal trajectory in both state and control variable to minimize the cost functional

[1]Corresponding author: ***Olotu, O.,*** E-mail: segolotu@yahoo.ca , Tel. +2348062706593

$$J = \int_{t_0}^{t_f} L(t, x(t), u(t))dt \tag{1}$$

Subject to: $\dot{x}(t) = f(t, x(t), u(t)), \quad t \in \left[t_0, t_f\right]$ (2)

where x, u are state and control variables respectively.

Methods for solving optimal control problems can be divided into two basic classes: indirect and direct methods. In the indirect approach, the optimal control problem is transformed into a boundary value problem by formulating the first order necessary conditions for optimality, thereby obtaining the Euler –Lagrange system [3], [4], [5]. The resulting 2-point boundary value differential-algebraic equation is solved analytically by standard differential techniques through the formulation of Riccati equation [2]. In the direct approach, the optimal control problem is approximated by a parameter optimization problem in which the first order optimality conditions are not explicitly included. Earlier Authors discretized the performance index and constraint using rectangular and Euler schemes respectively and obtained an unconstrained formulation by adjoining the objective and constraint using exterior penalty function method. Now, a nonlinear programming algorithm with Conjugate gradient method (CGM) is used to obtain solution to (1) and (2).

3.0 Method of Solution

In this work, Augmented Lagrangian method is described in a similar procedure as exterior penalty technique.
Consider Optimal Control Problem of the form,

$$\min J(x,u) = \int_0^Z (px^2(t) + qu^2(t))dt \tag{3}$$

Subject to: $\dot{x} = ax(t) + bu(t)$ (4)
$x(t_0) = c, \;\; t \in [0, Z]$

where a, b are real constants, $p, q > 0$, $c \in R$

In order to solve problem (3) and (4) by CGM, we replace the constrained problem by an appropriate approximate discretized optimal control problem [6]. $i.e$ breaking the interval into n equal subintervals with knots $0 = t_0 < t_1 < t_2 < \cdots < t_n = Z$ and $\Delta t_k = 0.01$ and $t_k = k\Delta t_k, k = 0,1,2, \dots, n-1, n$

Discretising equation (3) using trapezoidal rule and (4) using crank-Nicholson we have,

$$\int_0^Z (px^2(t) + qu^2(t))dt = \frac{h}{2}\sum_{k=1}^{N}[\,p(x^2(t_k) + x^2(t_{k-1})) + q(u^2(t_k) + u^2(t_{k-1}))] \tag{5}$$

where $h = \dfrac{Z}{N}$

Writing this in matrix form, we have,

$$\begin{pmatrix} x_1 & x_2 & \cdots & x_N & u_0 & u_1 & \cdots & u_N \end{pmatrix} \begin{pmatrix} ph & & & & & & & \\ & ph & & & & & & \\ & & \ddots & & & & & \\ & & & p\frac{h}{2} & & & & \\ & & & & q\frac{h}{2} & & & \\ & & & & & qh & & \\ & & & & & & \ddots & \\ & & & & & & & q\frac{h}{2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \\ u_0 \\ u_1 \\ \vdots \\ u_N \end{pmatrix} + C_0 p\frac{h}{2} \tag{6}$$

This can be re-written as

$$V^T AV + C', \tag{7a}$$

where,

$$V^T = \begin{pmatrix} x_1 & x_2 & x_3 & \cdots & x_N & u_0 & u_1 & u_2 & \cdots & u_N \end{pmatrix} \tag{7b}$$

$$A_{ii} = \begin{cases} ph, & i = 1,2,3,\dots N-1 \\ p\frac{h}{2}, & i = N \\ q\frac{h}{2}, & i = N+1 \\ qh, & i = N+2, N+2, \dots .2N \\ q\frac{h}{2}, & i = 2N+1 \end{cases} \tag{7c}$$

and

$$C = C_0 p \frac{h}{2} \tag{7d}$$

We seek to discretize our constraint using second order one step implicit trapezoidal rule (Crank-Nicholson) [7].

$$\dot{x}(t_k) = \frac{x_{k+1} - x_k}{h} = f(x_{k+\frac{1}{2}}, u_{k+\frac{1}{2}}) = \frac{1}{2}\{ f(x_{k+1}, u_{k+1}) + f(x_k, u_k)\} + O(h^2) \tag{8}$$

$$x_{k+1} - x_k = \frac{h}{2}[f(x_{k+1}, u_{k+1}) + f(x_k, u_k)] + O(h^3)$$

$$(1 - a\frac{h}{2})x_{k+1} = (1 + a\frac{h}{2})x_k + b\frac{h}{2}u_{k+1} + b\frac{h}{2}u_k$$

$$x_{k+1} = \bar{a}x_k + \bar{b}u_{k+1} + \bar{b}u_k \tag{9}$$

Where, $\bar{a} = \frac{(2+ah)}{(2-ah)}$ and $\bar{b} = \frac{bh}{(2-ah)}$ (10)

Hence, the discretized dynamical system becomes

$$x_{k+1} = \bar{a}x_k + \bar{b}u_{k+1} + \bar{b}u_k \tag{11}$$

$$\left(\begin{array}{cccccc:ccccccc} 1 & & & & & & -\bar{b} & -\bar{b} & & & & & \\ -\bar{a} & 1 & & & & & 0 & -\bar{b} & -\bar{b} & & & & \\ 0 & -\bar{a} & 1 & & & & 0 & 0 & -\bar{b} & -\bar{b} & & & \\ 0 & 0 & -\bar{a} & 1 & & & 0 & 0 & 0 & -\bar{b} & -\bar{b} & & \\ \vdots & & 0 & -\bar{a} & 1 & & \vdots & & & 0 & -\bar{b} & -\bar{b} & \\ 0 & \cdots & 0 & 0 & -\bar{a} & 1 & 0 & 0 & 0 & \cdots & 0 & -\bar{b} & -\bar{b} \end{array}\right)\begin{pmatrix} x_1 \\ x_2 \\ \\ \vdots \\ x_N \\ u_0 \\ u_1 \\ u_2 \\ \\ \vdots \\ u_N \end{pmatrix} = \begin{pmatrix} c_0 \\ 0 \\ 0 \\ \\ \vdots \\ 0 \end{pmatrix} \tag{12}$$

Eequation (12) can be written as

$$\left(E \vdots F\right)\left(V\right) = K$$

Where E is a bidiagonal matrix, F is also a bidiagonal matrix but V and K are column vectors respectively. Where $J = \left(E \vdots F\right)$ and $JV = K$

Where J is of dimension $N \times (2N+1)$, V is of dimension $(2N+1) \times 1$, and $K = (N \times 1)$

Therefore, the discretised optimal control problem becomes,

$$\frac{h}{2}\sum_{k=1}^{N}[p(x^2(t_k)+x^2(t_{k-1}))+q(u^2(t_k)+u^2(t_{k-1}))] \tag{13}$$

Subject to

$$x_{k+1} = \bar{a}x_k + \bar{b}u_{k+1} + \bar{b}u_k \tag{14}$$

By parameter optimization [8], the discretised optimal control becomes

$$V^T AV + C \tag{15}$$

Subject to

$$JV = K \tag{16}$$

V is a column vector of dimension $(2N+1) \times 1$, $V^T = (x_1, x_2, x_2, \cdots, x_N, u_0, u_1, u_2, \cdots u_N)$ and

A is a square matrix of dimension $(2N+1) by (2N+1)$.

Starting from 1968, a number of Researchers have proposed a new class of methods, called methods of multiplier in which the penalty idea is merged with the primal-dual and Lagrangian philosophy. In the original method of multiplier (Augmented Lagrangian method), proposed by Hestenes and Powell [10] the quadratic penalty term is added not only to the objective function $(Z^T AZ + C)$ of (ECP) but rather to the Lagrangian function [9]

$$L = V^T AV + C + \lambda^T (JV - K) \tag{17}$$

Hence, the Augmented Lagrangian function from equation (16) becomes

Minimize $$L_p(v,\mu,\lambda) = V^T AV + \lambda^T |JV - K| + \frac{1}{\mu}\|JV - K\|^2 \tag{18}$$

On expansion we have,

$$L_p = V^T\left(A + (\tfrac{1}{\mu}J^T J)\right)V + \left(\lambda^T J - \tfrac{2}{\mu}K^T J\right)V + (\tfrac{1}{\mu}K^T K - \lambda^T K + C') \tag{19}$$

$$L_p = V^T A_p V + B^T V + C \tag{20}$$

where B^T is of dimension $1 \times (2N+1)$
A_p is of dimension $(2N+1) \times (2N+1)$
C is of dimension 1×1

This equation (21) is a quadratic programming problem which can be solved via Conjugate Gradient Method (CGM)

where $$A_p = A + \frac{1}{\mu} J^T J \tag{21}$$

$$B^T = \lambda^T J - \frac{2}{\mu} K^T J \tag{22}$$

$$D = \frac{1}{\mu} K^T K - \lambda^T K + C \tag{23}$$

***Lemma* 3.1**: Consider the formulated quadratic function (18), the penalized matrix $A_\rho = \left[A + \frac{1}{\mu} J^T J \right]$ is said to be positive definite.

Proof: See [2].

The positive definiteness of the penalized matrix makes the scheme amenable to conjugate gradient method. We solve the unconstrained minimization equation (20) by conjugate gradient algorithm in the inner loop and enforce the feasibility condition in the outer loops as stated in the algorithm below.

4.0 Algorithm For The Scheme

(i) Choose $V_{0,0} \epsilon\ ^{2N+1}, C > 0, \mu > 0, \lambda > 0, d > 0. Set\ j = 0$

(ii) Set $i = 0\ and\ p_0 = -g_0 = -\nabla L_p(V_{0,0})$

(iii) Compute $\alpha_i = \frac{g_i^T g_i}{p_i^T A p_i}$

(iv) Set $V_{j,i+1} = V_{j,i} + \alpha_i p_i$

(v) Compute $\nabla L_p(V_{j,i+1})$

(vi) If $\nabla L_p(v_{j,i+1}) = 0\ and\ JV_{j,i+1} = K, Stop\ else\ go\ to$ (vii)

(vii) If $\nabla L_p\ (v_{j,i+1}) \neq 0, set\ g_{i+1} = \nabla L_p(V_{j,i+1})$

$$p_{i+1} = -g_{i+1} + \gamma_i p_i$$

$$\gamma_i = \frac{g_{i+1}^T g_{i+1}}{g_i^T g_i}$$

$(viii)$ Set $i = i + 1\ and\ go\ to\ step\ 3$

$(i\times)$ Else, if $JV_{j,i+1} \neq K$ ***or*** $JV_{j,i+1} - K = 0, then$

set $\mu_{k+1} = d\mu_k$

$\lambda_{j+1} = [\lambda_j + \mu_j (JV - K)$

$(\times)$ Set $j = j + 1\ and\ go\ to\ step\ (2)$

5.0 Numerical Examples and Presentation of Results.

Example 5.1. $Consider\ the\ optimal\ control\ problem$

$$\min I(x,u) = \int_0^1 (x^2(t) + u^2(t))dt \tag{24}$$

Subject to:

$$\dot{x}(t) = 1.705x(t) + 3.021u(t), x(0) = 1 \quad (25)$$

where $p = 1,\ q = 1,\ a = 1.705,\ b = 3.021$

We now present the results of the investigations based on the operator (A_p). The results presented in Table 1 shows the accuracy and the efficiency of using Augmented Lagrangian function on the discretised optimal control problem amenable to conjugate gradient method compared to exterior penalty function, taking $\mu = 1000, h = 0.01$ for both schemes.

Table 1. Comparison between existing method and the newly developed scheme.

Iterations	Constraints Satisfaction		Objective Value	
	DCAQP (2011)	New Scheme	DCAQP (2011)	New Scheme
1	2.1191E-3	2.0933E-3	0.5700	0.5605
2	2.1381E-4	1.0661E-4	0.5741	0.5648
3	2.1400E-5	5.3406E-6	0.5746	0.5649
4	2.1402E-6	2.6708E-7	0.5746	0.5649
5	2.1402E-7	1.3335E-8	0.5746	0.5649
6	2.1402E-8		0.5746	

By [2], the analytical solution is

$$\begin{pmatrix} x \\ \mu \end{pmatrix} = \begin{pmatrix} 0.0028 \\ 0.0010 \end{pmatrix} e^{3.4689t} + \begin{pmatrix} 0.9971 \\ -1.1305 \end{pmatrix} e^{-3.4689t}, \quad (26)$$

Control variable is $u(t) = 0.0015e^{3.4689t} - 1.7076e^{-3.4689t}$. (27)

The analytical objective function value is $I = 0.5647$ and the objective value using exterior penalty method is $I = 0.5746$ while the objective value using augmented lagrangian is $I = 0.5649$

Example 5.2. *Consider the optimal control problem*

$$\min \quad I(x,u) = \int_0^1 (x^2(t) + u^2(t))dt \quad (28)$$

Subject to

$$\dot{x} = 2x(t) + 5u(t), \qquad x(0) = 1 \quad (29)$$

where $p = 1,\ q = 1,\ a = 2,\ b = 5$

By [2], the analytical solution is

$$\begin{pmatrix} x \\ \mu \end{pmatrix} = \begin{pmatrix} 1.0000 \\ -0.5908 \end{pmatrix} e^{-5.3852t} \quad (30)$$

The control variable is given as,

$$u(t) = -1.4770e^{-5.3852t}. \quad (31)$$

The analytic objective function value is $I = 0.2954$ and the objective value using exterior penalty method amenable to conjugate gradient is $I = 0.3024$ while the objective function value using Augmented Lagrangian amenable to conjugate gradient is $I = 0.2956$ as we can see in the Table 2.

Table 2. **Comparison of results using existing scheme and the developed scheme.**

Iterations	Constraints Satisfaction		Objective Value	
	DCAQP (2011)	New Scheme	DCAQP (2011)	New Scheme
1	0.6697E-1	0.6633E-1	0.2430	0.2366
2	0.8729E-2	0.5414E-2	0.2955	0.2926
3	0.9010E-3	0.2891E-3	0.3026	0.2955
4	0.9039E-4	0.1455E-4	0.3033	0.2956
5	0.9042E-5	0.7282E-6	0.3034	0.2956
6	0.9042E-6	0.3641E-7	0.3034	0.2956
7	0.9042E-7		0.3034	

6.0. Convergence Analysis

Naturally, solving an approximate problem, we can only be expected to obtain an approximate solution of the original problem. In this research, we construct a sequence of approximate problems which converges in a well-defined sense to the original problem with some error of tolerance. Then the corresponding sequence of approximations yields in the limit, a solution of the original problem.

Considering the algorithm in section 4 above, we are only concerned with the speed at which the algorithm converges to a limit.

Given a sequence

$$\{z_k\} \subset R^{2n+1} \quad \text{with} \quad z_k \to z^*$$

The typical approach is to measure the speed (rate) of convergence in terms of error function.

$$e : R^{2n+1} \to R$$

Satisfying $e(z) \geq 0$ for all $z \in R^{2n+1}$ and $e(z^*) = 0$

Where $e(z) = \left| z - z^* \right|$

Suppose

$$e_k \neq 0 \quad \forall\, k\,,$$

Our convergence ratio (β) becomes

$$\beta = \lim_{k\to\infty} \frac{e_{k+1}}{{e_k}^2} \quad \text{Or} \quad \beta = \lim_{k\to\infty} \frac{\left\| Z_k - Z^* \right\|}{(\left\| Z_{k-1} - Z^* \right\|)^2}\ [10,11].$$

If $0 < \beta < 1$, then $\{z_k\}$ converges quadratically with convergence ratio β. If $\beta = 0$, then $\{z_k\}$ converges super-linearly. If $\beta = 1$, then $\{z_k\}$ converges sublinearly.

Though optimization algorithms are known or expected to converge linearly, quadratically or super linearly, however, quadratic convergence is the most satisfactory for optimization algorithms provided the convergence ratio is not close to one as we can see in example 5.1 as shown in Table 3.

Table 3. Convergence ratio of the iterations

Penalty Parameter(μ)	**Objective Value (r)**	**Convergence ratio (β)**
1.0×10^{-3}	0.5605	0.0087
1.0×10^{-4}	0.5648	0.3711
1.0×10^{-5}	0.5649	0.6363
1.0×10^{-6}	0.5649	0.6447
1.0×10^{-7}	0.5649	0.6450

7.0. Conclusion and Comments

We have shown that Discrete Optimal Control problem can be solved via Conjugate Gradient Method using exterior penalty method and augmented Lagrangian method to construct the control operator (A_ρ). The solutions of both methods compare favourably to the analytical solution. However, it is observed that the new scheme agrees better to the exact solution in terms of accuracy and convergence. Hence, it is a better scheme.

References

[1] Pablo, P. (2004), Introduction to Optimization. Springer-Verlag, New York.
[2] Adekunle, A.I (2011), Algorithm for a class of discretised constrained optimal control problems.M.Tech Thesis, Federal University of Technology, Akure, Ondo state, Nigeria.
[3] Ross, I.M (2009), A Primer on pontryagin's principle in optimal control, collegiate publisher
[4] Bryson, A.E. and Ho, Y.C (1975), Applied Optimal control, Hemisphere publishing company.
[5] Hestenes, M. R. and Stiefel, E. (1952), Methods of Conjugate Gradients for Solving Linear Systems. Institute of Research of the National Burean of Standards, Vol. 49, 409436.
[6] Olotu, O. and Olorunsola, S.A. (2006), An Algorithm for a Discretized Constrained, Continuous Quadratic Control Problem. Journal of Applied Sciences, Vol.9 (1), 6249-6260.
[7] Burlirsch, R. Stoer, J. (1993), Introduction to Numerical Analysis. Springer-Verlag, New York, second edition.
[8] Betts, J.T (2001), Practical Methods for Optimal Control Problem Using Nonlinear Programming. SIAM, Philadelphia.
[9] Rockafellar, R.T. (1973), The multiplier Method of Hestenes and Powell Applied to Convex programming, Journal of Optimization Theory and Application.
[10] Bertesekas, D.P (1996), Constrained Optimization and Lagrange multiplier methods. Athena Scientific Belmont, Massachussets.
[11] Olotu, O. and Olorunsola, S.A. (2009), Convergence profile of a Discretized Scheme for Constrained Problem via the Penalty-Multiplier Method. Journal of the Nigerian Association of Mathematical Physics, Vol. 14(1), 341-348.