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On The Algorithm for Dynamic Restoring Control Problems with Matrix Coefficients

[1]Olotu, O. and [2]Olorunsola, S.A

[1]The Federal University of Technology, Mathematical Sciences Department, P.M.B.704, Akure, Ondo State, Nigeria
[2]University of Ado-Ekiti, Mathematical Sciences Department, P.M.b. 5363, Ado-Ekiti, Ekiti State, Nigertia.

Abstract

An algorithm is hereby developed to solve a class of control problems constrained by dynamic restoring type with matrix coefficients numerically. The penalty-multiplier method is evolved to obtain an unconstrained discretized formulation. With the bilinear form expression, an associated operator is constructed via a theorem to circumvent the cumbersomeness inherent in some earlier methods; particularly the Function space algorithm (FSA).The conjugate gradient method (CGM) is evoked to solve the discretized problem. One sampled problem is solved numerically and the convergence analysis is found to be linearly convergent as demonstrated in the output data tables.

Keywords: Penalty-Multiplier, matrix coefficients, operator and linear convergence.

1.0 Introduction

Developing an efficient Truly algorithm for solving optimization problems is a field of Research that is evolving very rapidly, so that the methods seeming best today will probably be abandoned soon and replaced either by old ideas in a new framework or by entirely novel techniques . Hence, we are developing a new algorithm using the Augmented Lagrangian Method to obtain an approximate problem of the original problem. Naturally, if we construct a sequence of approximate problems converging, in a well defined sense to the original problem, then expectedly the corresponding sequence of approximate solutions will give a solution of the original problem.

Here, the analytic method of solution is not available having failed to satisfy the necessary optimality conditions due to the presence of the delay and advanced terms. Therefore, this algorithm examines the approximate solutions with the limiting solution. Using the finite difference method for its differential constraint, discretization of its time interval and with the application of the penalty and Multiplier methods [1], an unconstrained discretized formulation of the problem is obtained. With this formulation, a bilinear form expression of the problem is recast. which forms a framework for the construction of an operator based on Modified Ibiejugba's reviewed method[2] on function minimization by Fletcher and Reevesl[3] . A sequence of estimates is generated by conjugate gradient method (CGM) with stepsize formulated by the minimization rule. Finally, these estimates are examined for linear convergence through a convergence analysis scheme[4]. An hypothetical constrained problem with vector-matrix coefficients are examined to test the efficiency of the developed scheme for solution and convergence profile of the objective function values. In the next section, a general quadratic control problem is considered for our developed scheme.

2.0 Generalized Problem

$$Minimize\, J(t,\mu,x,u)_K = \int_0^Z (x(t)^T Px(t) + u(t)^T Qu(t))dt \qquad (2.1)$$

such that

$$\dot{x}(t) = Ax(t) + Bx(t-r) + Cu(t), \quad x(0) = x_0 \quad \dot{x}(t) = h(t), t \in [-r, 0] \qquad (2.2)$$

[1]Corresponding author: ***Olotu, O.,*** E-mail: segolotu@yahoo.ca , Tel. +2348033509040

where $x(t),\ u(t)\varepsilon R^2$, P,Q are 2×2 square symmetric matrices with A, B and C as 2×2 matrices not necessarily symmetric and K denotes the Cartesian product of the following two spaces $K = H_1[0,\sigma]\ x\ L_2^q[0,\sigma]$, where $H_1[0,\sigma]$ stands for the Sobolev space of the absolutely continuous functions $x(\bullet)$ and $\dot{x}(\bullet)$ are square integrable over $[0,\sigma]$ and $L_2^q[0,\sigma]$ stands for the Hilbert space consisting of equivalence classes of square integrable functions from $[0,\sigma]$ to R^q.

3.0 Discretization

In this section, we shall convert the constrained continuous problem in to un constrained discretized problem using discertization for its time interval and finite difference method for the constraint respectively.

By discretizing (2.2), using [1,4], subdivide [0, Z] into n equal intervals $[t_k,t_{k+1}]$ at mesh points $x_0<x_1<x_2<x_3<\ldots<x_{n-1}<x_n$, where n-1 is the number of partition points chosen arbitrarily, thus having (n+1) partition points, with $x_j=j\Delta j$, j=0,1,2,…,n, and $\Delta j=\Delta k$ is the fixed length of each subinterval for j=k.

Let $t_0 = 0$ and $t_k = \sum_{j=1}^{k-1}\Delta j,\quad k=1,2,3,\ldots,n,\quad t_n = Z$, $x(k) = x_k(t_{k,}),\ u(k) = u_k(t_k)$, k=0,1,…,n.

By Euler's scheme or finite difference method ,

$$\dot{x}(k) = (x(k+1) - x(k))/\Delta k \qquad k=0,1,2,\ldots..N-1$$

$$\dot{x}_k(t_k) = Ax_k(t_k) + Bx_k(t_k - r_k) + Cu_k(t_k)$$

$$\left(x(k+1) - x(k)\right)/\Delta_k = Ax_k(t_k) + Bx_k(t_k - r_k) + Cu_k(t_k)$$

$$\langle \lambda_k, \dot{x}(t_k) - Ax_k(t_k) - Bx_k(t_k - r_k) - Cu_k(t_k)\rangle,\quad x_0(0) = 0 \tag{3.1}$$

We then have the generalized problem (2.1) in the form;

$$M\text{ in } J_{(x,u)} = \sum_{k=0}^{n}\Delta_k\left(x_k(t_k)^T P x_k(t_k) + u_k(t_k)^T Q u_k(t_k)\right) \tag{3.2}$$

subject to

$$\begin{aligned}&\left(x_{k+1}(t_{k+1}) - x_k(t_k)\right)/\Delta_k = Ax_k(t_k) + Bx_k(t_k - r_k) + Cu_k(t_k)\\ &x_k(0) = 0\end{aligned} \tag{3.3}$$

4.0 Application Of The Penalty And Multiplier Parameters.

Here, the constrained problem is converted into an unconstrained approximated problem to eliminate some or all of the constraints and add to the objective function a term which prescribes a high cost to infeasibility points.

Applying the penalty function and the multiplier method [5,7,8] to (3.2)-(3.3), we obtain after expanding and collecting like-terms,

$$\begin{aligned}M\text{in} J(x_k,u_k) = \sum_{k=0}^{n}\{&x_k(t_k)^T\alpha_k x_k(t_k) + u_k(t_k)^T\beta_k u_k(t_k) + y_k(t_k)^T\mu y_k(t_k)\\ &+ x_k(t_k)^T y_k(t_k)c_k + x_k(t_k - r_k)^T y_k(t_k)n_k + u_k(t_k)^T y_k(t_k)mk1\\ &+ x_k(t_k - r_k)^T x_k(t_k - r_k)mk + u_k(t_k)^T x_k(t_k)mk2 + x_k(t_k)^T x_k(t_k - r_k)mk3\\ &+ u_k(t_k)^T x_k(t_k - r_k)mk4 + \lambda_k(t_k)^T y_k(t_k) - \lambda_k(t_k)^T x_k(t_k) - \lambda_k(t_k)^T\Delta_k A u_k(t_k)\\ &- \lambda_k(t_k)^T\Delta_k A x_k(t_k) - \lambda_k(t_k)^T\Delta_k B x_k(t_k - r_k) - \lambda_k(t_k)^T\Delta_k C u_k(t_k)\}\end{aligned} \tag{4.1}$$

where $y_k(t_k) = x_{k+1}(t_{k+1})$, $\alpha_k = \mu + 2\mu\Delta_k A + \Delta_k{}^2 A^T A\mu + P\Delta_k$, $\beta_k = Q\Delta_k + \Delta_k{}^2 C^T C\mu$, $c_k = -2\mu - 2\mu\Delta_k A^T$, $n_k = -2\mu\Delta_k B^T$, $m_k = \mu\Delta_k{}^2 B^T B$, $m_{k1} = -2\Delta_k C\mu$, $m_{k2} = 2\mu\Delta_k C + 2\mu\Delta_k{}^2 C^T A$, $m_{k3} = 2\mu\Delta_k{}^2 B^T A + 2\mu\Delta_k B$, $m_{k4} = 2\mu\Delta_k{}^2 C^T B$.

5.0 Construct of Operator V

We now formulate the bilinear form expression from (4.1) serving as framework for the construction

$$\begin{aligned}\langle Z_{k1}(t_k), VZ_{k2}(t_k)\rangle = \sum_{k=0}^{n}\{&\alpha_k x_{k1}(t_k)^T x_{k2}(t_k) + \beta_k u_{k1}(t_k)^T u_{k2}(t_k) \\ &+ y_{k1}(t_k)^T y_{k2}(t_k)\mu + x_{k1}(t_k)^T y_{k2}(t_k)c_k + x_{k2}(t_k)^T y_{k1}(t_k)c_k \\ &+ x_{k1}(t_k - r_k)^T y_{k2}(t_k)n_k + x_{k2}(t_k - r_k)^T y_{k1}(t_k)n_k + u_{k1}(t_k)^T y_{k2}(t_k)m_{k1} \\ &+ u_{k2}(t_k)^T y_{k1}(t_k)m_{k1} + x_{k1}(t_k - r_k)^T x_{k2}(t_k - r_k)m_k + u_{k1}(t_k)^T x_{k2}(t_k)m_{k2} \\ &+ u_{k2}(t_k)^T x_{k1}(t_k)m_{k2} + x_{k1}(t_k)^T x_{k2}(t_k - r_k)m_{k3} + x_{k2}(t_k)^T x_{k1}(t_k - r_k)m_{k3} \\ &+ u_{k1}(t_k)^T x_{k2}(t_k - r_k)m_{k4} + u_{k2}(t_k)^T x_{k1}(t_k - r_k)m_{k4} \\ &+ \lambda_{k1}(t_k)^T y_{k2}(t_k) + \lambda_{k2}(t_k)^T y_{k1}(t_k) - \lambda_{k1}(t_k)^T x_{k2}(t_k) - \lambda_{k2}(t_k)^T x_{k1}(t_k) \\ &- \lambda_{k1}(t_k)^T x_{k2}(t_k)\Delta_k A - \lambda_{k2}(t_k)^T x_{k1}(t_k)\Delta_k A - \lambda_{k1}(t_k)^T x_{k2}(t_k - r_k)\Delta_k B \\ &- \lambda_{k2}(t_k)^T x_{k1}(t_k - r_k)\Delta_k B - \lambda_{k1}(t_k)^T u_{k2}(t_k)\Delta_k C - \lambda_{k2}(t_k)^T u_{k1}(t_k)\Delta_k C\end{aligned} \tag{5.1}$$

And the operator is constructed thus;
Letting,

$$VZ_{K2}(t_k) = \begin{pmatrix} V_{11} & V_{12} & V_{13} & V_{14} \\ V_{21} & V_{22} & V_{23} & V_{24} \\ V_{31} & V_{32} & V_{33} & V_{34} \\ V_{41} & V_{42} & V_{43} & V_{44} \end{pmatrix}\begin{pmatrix} x_{k2}(t_k) \\ u_{k2}(t_k) \\ h_{k2}(t_k) \\ \lambda_{k2}(t_k)\end{pmatrix} = \begin{pmatrix} V_{11} \\ V_{21} \\ V_{31} \\ V_{41}\end{pmatrix} \tag{5.2}$$

where $z_k(t_k) = (x_k(t_k), u_k(t_k), h_k(t_k), \lambda_k(t_k))$.
Using Euler's scheme and simplifying (5.1), we have the following governing equation

$$\begin{aligned}\langle Z_{K1}(t_k), VZ_{K2}(t_k)\rangle_H = \sum_{k=0}^{n}\{&\alpha_k x_{K1}(t_k)^T x_{K2}(t_k) + \beta_k u_{K1}(t_k)^T u_{K2}(t_k) \\ &+ x_{K1}(t_k - r_k)^T x_{K2}(t_k - r_k)m_k + x_{K1}(t_k)^T x_{K2}(t_k)\mu \\ &+ x_{K1}(t_k)^T \dot{x}_{K2}(t_k)\Delta_k\mu + \dot{x}_{K1}(t_k)^T x_{K2}(t_k)\Delta_k\mu \\ &+ \mu\Delta_K{}^2\dot{x}_{K1}(t_k)^T \dot{x}_{K2}(t_k) + x_{K1}(t_k)^T x_{K2}c_K + x_{K1}(t_k)^T \dot{x}_{K2}(t_k)\Delta_K c_k \\ &+ x_{K2}(t_k)^T \dot{x}_{K1}\Delta_k c_K + x_{K1}(t_k - r_k)^T x_{K2}(t_k)n_k \\ &+ x_{K1}(t_k - r_k)^T \dot{x}_{K2}(t_k)\Delta_k n_k + x_{K2}(t_k - r_k)^T x_{K1}(t_k)n_k \\ &+ x_{K2}(t_k - r_k)^T \dot{x}_{K1}(t_k)\Delta_k n_k + u_{K1}(t_k)^T x_{K2}(t_k)m_{k1} + u_{K1}(t_k)^T \dot{x}_{K2}(t_k)m_{k1}\end{aligned} \tag{5.3}$$

From (5.3) following[9], we initiate the next step as a theorem for establishing the operator V:
THEOREM 1
Let the initial guess of the conjugate algorithm be $Z_0(t_k)$ so that
$Z_0{}^T(t_k) = (x_0, u_0, h_o, \lambda_0)$.
Then the control operator V associated with the generalized problem (2.1) satisfying Vz_{k2} is given by
$Z_{k2}(t_k) = (x_{k2}(t_k), u_{k2}(t_k), h_{k2}(t_k), \lambda_{k2}(t_k))$.

PROOF OF THEOREM 1

Solve for $x_{k2}(t)$ by setting $u_{k2}(t_k)=h_{k2}(t_k)=\lambda_{k2}(t_k)=0$ in (5.3) and collecting like-terms, we have

$$\begin{aligned}\langle Z_{K1}(t_k),VZ_{K2}(t_k)\rangle_H &= \sum_{k=0}^{n}\{x_{k1}{}^T(t_k)[x_{k2}(t_k)(\alpha_k+\mu+2c_k+n_k+m_{k3})\\ &+\dot{x}_{k2}(t_k)^T(\mu\Delta_k+\Delta_k c_k)+x_{k2}(t_k+r_k)(n_k+m_{k3}+m_k)+\dot{x}_{k2}(t_k+r_k)n_k\Delta_k]\\ &+\dot{x}_{k1}{}^T(t_k)[x_{k2}(t_k)(\Delta_k(n_k+\mu+c_k)+\dot{x}_{k2}(t_k).\mu\Delta_k{}^2]+u_{k1}{}^T(t_k)[x_{k2}(t_k)(m_k+m_{k2}+m_{k4})\\ &+\dot{x}_{k2}(t_k).\Delta_k m_{k1}]+h_{k1}{}^T.[x_{k2}(t_k+r_k)(n_k+m_{k3}+m_k)+\dot{x}_{k2}(t_k+r_k)n_k\Delta_k]\\ &+\lambda_{k1}(t_k)^T[x_{k2}(t_k)(-\Delta_k(A+B))+\dot{x}_{k2}(t_k)\Delta_k]\end{aligned} \tag{5.4}$$

$$\langle Z_{K1}(t_k),VZ_{K2}(t_k)\rangle_H = \sum_{k=0}^{n}\{x_{k1}(t_k)^T V_{11}+\dot{x}_{k1}(t_k)^T\dot{V}_{11}.+u_{k1}(t_k)^T V_{21}+h_{k1}(t_k)^T V_{31}+\lambda_{k1}(t_k)^T V_{41} \tag{5.5}$$

where

$$V_{41}(t_k)=x_{k2}(t_k)(-\Delta_k(A+B)+\dot{x}_{k2}(t_k)\Delta_k \tag{5.6}$$

$$V_{31}(t_k)=x_{k2}(t_k+r_k)(n_k+m_{k3}+m_k)+\dot{x}_{k2}(t_k+r_k)n_k\Delta_k \tag{5.7}$$

$$V_{21}(t_k)=x_{k2}(t_k)(\Delta_k(m_k+m_{k2}+m_{k4}))+\dot{x}_{k2}(t_k)\Delta_k m_{k1} \tag{5.8}$$

To determine $V_{11}(t_k)$, define

$$\begin{aligned}\Omega_1(t_k)&=x_{k2}(\alpha_k+\mu+2c_k+n_k+m_{k3})+\dot{x}_{k2}(t_k)(\Delta_k(\mu+c_k))+\dot{x}_{k2}(t_k+r_k)\Delta_k n_k\\ &+x_{k2}(t_k+r_k)(n_k+m_{k3}+m_k)\end{aligned} \tag{5.9a}$$

$$f_1(t_k)=x_{k2}(\Delta_k(\mu+n_k+c_k)+\dot{x}_{k2}(t_k).\mu\Delta_k{}^2 \tag{5.9b}$$

Now, $\Omega_1(t_k)$ and $f_1(t_k)$ are continuous functions on [0,Z], $V_{11}(t_k)$ is continuous and at least twice differentiable on [0,Z]. Hence $\Omega_1(t_k)-V_{11}(t_k)$ and $f_1(t_k)-\dot{V}_{11}(t_k)$ are continuous on [0,Z], $x(.)\in D_1[o,Z]$ such that $x(0)=x(Z)=0$ and

$$\int_0^Z\{x_{k1}(t_k)[\Omega_1(t_k)-V_{11}(t_k)]+\dot{x}_{k1}(t_k)[f_1(t_k)-\dot{V}_{11}(t_k)]\}dt_k=0 \tag{5.10}$$

Hence,

$$\frac{d}{dt_k}\left(f_1(t_k)-\dot{V}_{11}(t_k)\right)=\Omega_1(t_k)-V_{11}(t_k) \tag{5.11}$$

So

$$\dot{f}_1(t_k)-\ddot{V}_{11}(t_k)=\Omega_1(t_k)-V_{11}(t_k),\ 0\le t_k\le T. \tag{5.12}$$

Let

$$\ddot{V}_{11}(t_k)-V_{11}(t_k)=\dot{f}_1(t_k)-\Omega_1(t_k)=q(t_k) \tag{5.13}$$

This is a second order differential equation that needs to be solved. So we impose the following initial conditions;

$$V_{11}(0)=p_1\ and\ \dot{V}_{11}(0)=r_1 \tag{5.14}$$

where $p_1\ and\ r_1$ are to be determined.

Let $Q(s)=L(q(t_k))\ and\ V_{11}(s)=L(V_{11}(t_k))$ denote the Laplace transform of $q(t_k)$ and $V_{11}(t_k)$ respectively. Taking the Laplace transform of (5.13), we have,

$$s^2V_{11}(s)-p_1s-r_1-V_{11}(s)=Q(s) \tag{5.15}$$

$$V_{11}(s)=\frac{Q(s)}{s^2-1}+\frac{p_1s}{s^2-1}+\frac{r_1}{s^2-1} \tag{5.16}$$

We take the Laplace transform of (5.16) and use convolution theorem for the second term to obtain

$$V_{11}(t_k) = \int_0^Z q(s_k)\sinh(t_k - s_k)ds_k + p_1\cosh(t_k) + r_1\sinh(t_k) \tag{5.17}$$

But

$$\Omega_1(Z) - V_{11}(Z) = 0,\ \Omega_1(0) - V_{11}(0) = 0 \ and \ \Omega_1(0) = p_1 \tag{5.18}$$

So $V_{11}(0) = p_1$.

From (5.17) and (5.18),

$$V_{11}(T) = \int_0^T q(s_k)\sinh(T - s_k) + p_1\cosh(T) + r_1\sinh(T) \tag{5.19}$$

But $q(s_k) = \dot{f}_1(s_k) - \Omega_1(s_k)$ in (5.13). So, (5.17) becomes

$$\begin{aligned} V_{11}(t_k) &= -\sinh(Z) f_1(0) + \int_0^Z f_1(s_k)\cosh(t_k - s_k)ds_k \\ &\quad - \int_0^Z \Omega_1(s_k)\sinh(t_k - s_k)ds_k + p_1\cosh(t_k) + r_1\sinh(t_k) \end{aligned} \tag{5.20}$$

and

$$\begin{aligned} \tau_1 &= \frac{1}{\sinh(Z)}\{\sinh(Z) f_1(0) - \int_0^Z f_1(s_k)\cosh(Z - s_k)ds_k + \int_0^Z \Omega_1(s_k)\sinh(Z - s_k)ds_k \\ &\quad - \Omega_1(0)\cosh(Z) + \Omega_1(Z)\} \end{aligned} \tag{5.21}$$

Solve for $u_{k2}(t_k)$, by setting $x_{K2}(t_k) = h_{k2}(t_k) = \lambda_{k2}(t_k) = 0 \rightarrow \dot{x}_{k2}(t_k) = 0$ in (5.3) and collecting like-terms and following the same patterns as in equations (5.3) to (5.21), we have,

$$\begin{aligned} \min J(x_k, u_k) &= \sum_{k=0}^{n} u_{k1}(t_k)^T u_{k2}(t_k)\beta_k + u_{k2}(t_k)^T x_{k1}(t_k)m_{k1} + u_{k2}(t_k)^T \dot{x}_{k1}(t_k)\Delta_k m_{k1} \\ & u_{k2}(t_k)^T x_{k1}(t_k)\Delta_k m_{k2} + u_{k2}(t_k)^T x_{k1}(t_k - r_k)\Delta_k m_{k4} - \lambda_{k1}(t_k)^T u_{k2}(t_k)\Delta_k C \end{aligned} \tag{5.22}$$

Applying remark,

(i) $$x_{k1}(t_k - r_k) = x_{k1}(t_k) = \begin{cases} h_{k1}(t_k), & t_k \in [-r, 0] \\ x_{k1}(t_k), & t_k \in [0, Z - r] \end{cases}$$

to equation (5.22), we have

$$\begin{aligned} \min J(x_k, u_k) &= \sum_{k=0}^{n} x_{k1}{}^T(t_k)[u_{k2}(t_k)(m_{k1} + m_{k2}) + u_{k2}(t_k + r_k)m_{k4}] + \dot{x}_{k1}{}^T(t_k)[u_{k2}(t_k)\Delta_k m_{k1}] \\ & + u_{k1}{}^T(t_k)[u_{k2}(t_k)\beta_k h_{k1}{}^T(t_k)[u_{k2}(t_k + r_k)m_{k4}] + \lambda_{k1}{}^T(t_k)[u_{k2}(t_k)(-\Delta_k C)] \end{aligned} \tag{5.23}$$

$$\min J(x_k, u_k) = \sum_{k=0}^{n} x_{k1}(t_k)^T V_{12} + \dot{x}_{k1}(t_k)^T \dot{V}_{12} + u_{k1}(t_k)^T V_{22} + h_{k1}(t_k)^T V_{32} + \lambda_{k1}(t_k)^T V_{42} \tag{5.24}$$

where

$$V_{42}(t_k) = u_{k2}(t_k)^T(-\Delta_k C) \tag{5.25}$$

$$V_{32}(t_k) = u_{k2}(t_k + r_k)^T m_{k4} \tag{5.26}$$

$$V_{22}(t_k) = u_{k2}(t_k)^T \beta_k \tag{5.27}$$

$$\Omega_2(t_k) = u_{k2}(t_k)^T(m_{k1} + m_{k2}) + u_{k2}(t_k + r_k)^T m_{k4} \quad (5.28)$$

$$f_2(t_k) = u_{k2}{}^T(t_k)\Delta_k m_{k1} \quad (5.29)$$

$$V_{12}(t_k) = -\sinh(Z) f_2(0) + \int_0^Z f_2(s_k)\cosh(t_k - s_k)ds_k - \int_0^Z \Omega_2(s_k)\sinh(t_k - s_k)ds_k + p_2\cosh(t_k) + r_2\sinh(t_k) \quad (5.30)$$

where $r_2 = r_1$ with the exception that $f_1 \neq f_2$ and $\Omega_1 \neq \Omega_2$ as in equation (5.21).

Solve for $h_{k2}(t_k)$, by setting $x_{k2}(t_k) = u_{k2}(t_k) = \lambda_{k2}(t_k) = 0$, implying that $\dot{x}_{k2}(t_k) = 0$ in (5.3). By remark, setting $x_{k2}(t_k - r_k) = h_{k2}(t_k)$ after the above, collecting like-terms and following the same steps as in (5.3) to (5.21), we have

$$\min J(x_k, u_k) = \sum_{k=0}^{n} x_{k1}(t_k)^T[h_{k2}(t_k)(n_k + m_{k3} + m_k)] + \dot{x}_{k1}(t_k)^T[h_{k2}(t_k)\Delta_k n_k] + u_{k1}(t_k)^T[h_{k2}(t_k)m_{k4}] + h_{k1}(t_k)^T[h_{k2}(t_k)m_k] + \lambda_{k1}(t_k)^T h_{k2}(t_k)[-\Delta_k B] \quad (5.31)$$

$$\min J(x_k, u_k) = \sum_{k=0}^{n} x_{k1}(t_k)^T V_{13} + \dot{x}_{k1}(t_k)^T \dot{V}_{13} + u_{k1}(t_k)^T V_{23} + h_{k1}(t_k)^T V_{33} + \lambda_{k1}(t_k)^T V_{43} \quad (5.32)$$

Where,

$$V_{43}(t_k) = h_{k2}(t_k)^T[(-\Delta_k B)] \quad (5.33)$$

$$V_{33}(t_k) = h_{k2}(t_k)^T m_k \quad (5.34)$$

$$V_{23}(t_k) = h_{k2}(t_k)m_{k4} \quad (5.35)$$

$$V_{13}(t_k) = -\sinh(Z) f_3(0) + \int_0^Z f_3(s_k)\cosh(t_k - s_k)ds_k - \int_0^Z \Omega_3(t_k)\sinh(t_k - s_k)ds_k + p_3\cosh(t_k) + r_3\sinh(t_k) \quad (5.36)$$

where $r_3 = r_1$ except that $f_1 \neq f$ and $\Omega_1 \neq \Omega_3$ as in (5.21).

Solve for $\lambda_{k2}(t_k)$, by setting $x_{k2}(t_k) = u_{k2}(t_k) = h_{k2}(t_k) = 0$, implying that $\dot{x}_{k2}(t_k) = 0$ in (5.3). By remark, setting $x_{k2}(t_k - r_k) = x_{k2}(t_k)$ after the above, collecting like-terms and following the same steps as in (5.3) to (5.21), we have

$$\min J(x_k, u_k) = \sum_{k=0}^{n} x_{k1}(t_k)^T[\lambda_{k2}(t_k)(-\Delta_k(A+B))] + \dot{x}_{k1}(t_k)^T[\lambda_{k2}(t_k)\Delta_k] + u_{k1}(t_k)^T[\lambda_{k2}(t_k)(-\Delta_k] + h_{k1}(t_k)^T[\lambda_{k2}(t_k)(-\Delta_k)] + \lambda_{k1}(t_k)^T \quad (5.37)$$

$$\min J(x_k, u_k) = \sum_{k=0}^{n} x_{k1}(t_k)^T V_{14} + \dot{x}_{k1}(t_k)^T \dot{V}_{14} + u_{k1}(t_k)^T V_{24} + h_{k1}(t_k)^T V_{34} + \lambda_{k1}(t_k)^T V_{44} \quad (5.38)$$

where,

$$V_{44}(t_k) = 0 \quad (5.39)$$

$$V_{34}(t_k) = \lambda_{k2}(t_k)^T(-\Delta_k) \quad (5.40)$$

$$V_{24}(t_k) = \lambda_{k2}(t_k)^T(-\Delta_k) \quad (5.41)$$

$$\Omega_4(t_k) = \lambda_{k2}(t_k)^T(-\Delta_k(A+B)) \quad (5.42)$$

$$f_4(t_k) = \lambda_{k2}(t_k)\Delta_k \quad (5.43)$$

Having constructed operator V, written as

$$V = \begin{pmatrix} V_{11} & V_{12} & V_{13} & V_{14} \\ V_{21} & V_{22} & V_{23} & V_{24} \\ V_{31} & V_{32} & V_{33} & V_{34} \\ V_{41} & V_{42} & V_{43} & V_{44} \end{pmatrix} \qquad (5.44)$$

With this generalized scheme and associated operator, a program is written using the Conjugate Gradient Method Algorithm to execute and examine the convergence profile of the following sampled problem P1 with matrix coefficients. The result is tabulated in the following table for fixed penalty constant μ per cycle and updated Lagrangian multiplier. Please note that the state- control variables and Lagrangian multiplier are each 2-entry vector.

6.0 Data And Analysis

Example Problem P1

$$min\, J(,u) = \int_0^1 \left((x_1(t) + \frac{1}{2}x_2(t))^2 + \frac{3}{4}x_2{}^2(t) + 2u_1(t) + \frac{1}{2}u_2{}^2(t) + \frac{1}{2}u_2(t)) \right) dt$$

such that

$$\dot{x}_1(t) = x_1(t) - x_2(t) + x_1(t - 1/2) + 2x_2(t - 1/2) + 2u_1(t) + 2u_2(t)$$

$$\dot{x}_2(t) = x_1(t) + x_2(t) - x_1(t - .5) + x_2(t - .5) - u_2(t)$$

$$x_1(t) = 1 + t, \quad x_2(t) = t, \quad \forall t \in [-.5, 0]$$

$$x_1(0) = 1, \; x_2(0) = .5, \quad u_1(0) = 1, \quad u_2(0) = .5$$

$$x_0 = \begin{pmatrix} 1 \\ .5 \end{pmatrix}, \quad u_0 = \begin{pmatrix} 1 \\ .5 \end{pmatrix}, \quad \lambda(t) = \begin{pmatrix} 1+t \\ t \end{pmatrix}$$

$$where, \; P = \begin{pmatrix} 1 & .5 \\ .5 & 1 \end{pmatrix}, \quad Q = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix}$$

From the above problem, we obtain the following Tables 1 and 2 and Figures 1 and 2 showing the values of the objective functions per cycle; the convergence ratios profile; the state-control variables and the sinusoidal behaviour of the Lagrangian Multiplier parameter respectively.

TABLE 1 ; Objective Function Value Per Cycle

CYCLE NUMBER	PENALTY PARAMETER	OBJECTIVE FUNCTION VALUES	CONSTRAINT SATISFACTION
1	10^{-1}	0.6415	1.6530E-1
2	10^{-2}	1.1974	3.9512E-2
3	10^{-3}	1.3707	4.8877E-3
4	10^{-4}	1.3928	5.0150E-4
5	10^{-5}	1.3951	5.0282E-5
6	10^{-6}	1.3953	5.0296E-6
7	10^{-7}	1.3953	5.0297E-7

.

TABLE 2: Convergence Ratio Per Cycle.

CYCLE NUMBER	CONVERGENCE RATIO, $\left\| \frac{x_{k+1} - x^*}{x_k - x^*} \right\| < 1$
1	0.2397
2	0.1335
3	0.1034
4	0.0995.
5	0.0914
6	0.0826
7	0.0795

The following Figures, 1 and 2, show the state-control variables and the Lagrangian Multiplier- parameter respectively.

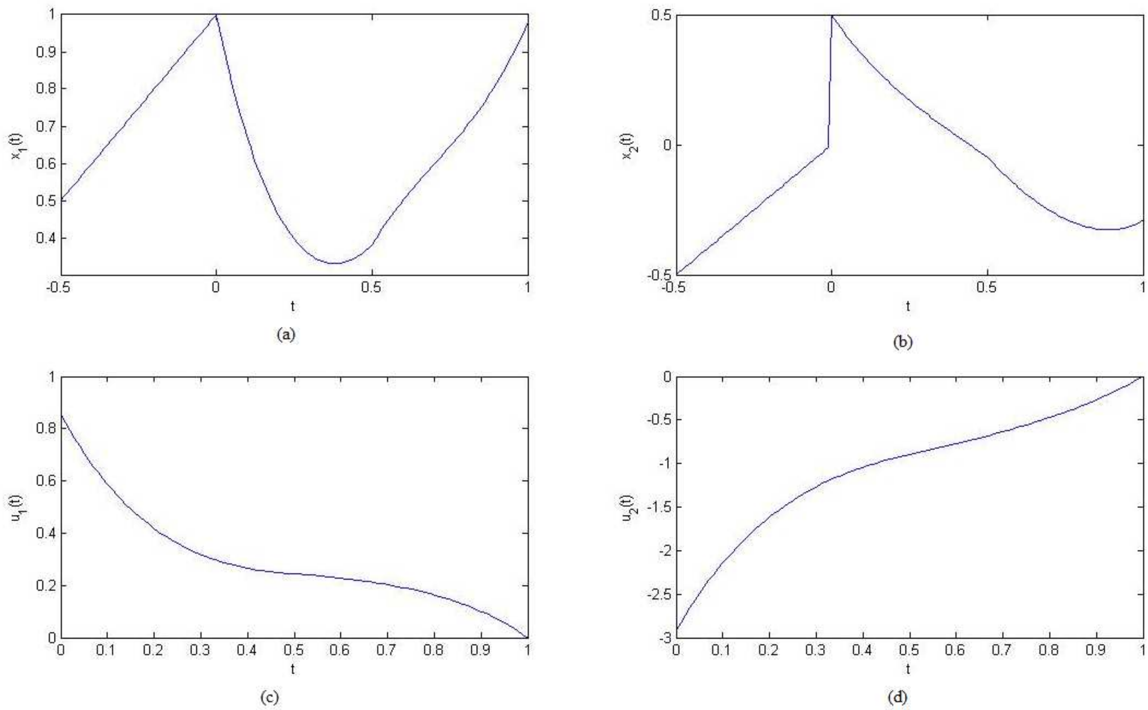
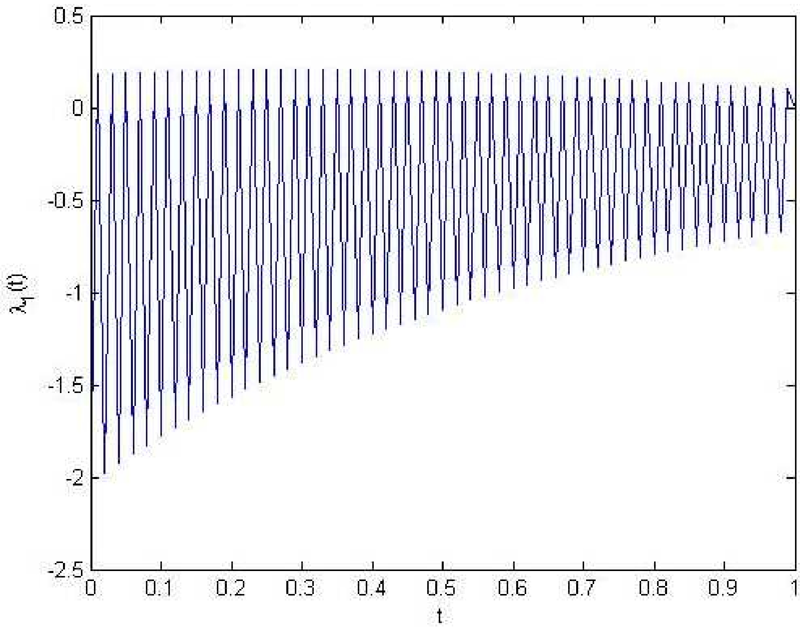
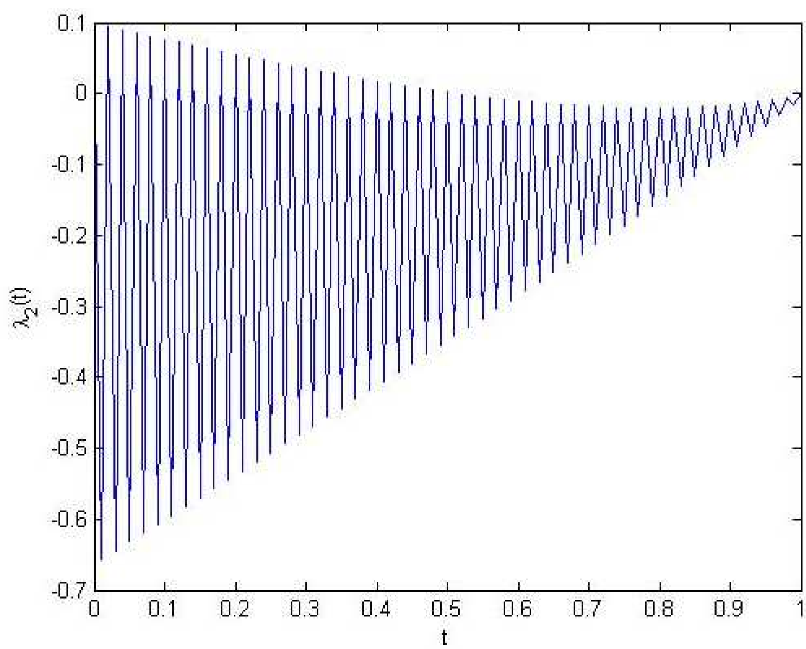


FIGURE 1(a, b, c and d): The variation of the state and control variables with respect to time t.



(a)



(b)

FIGURE 2(a and b) : The behaviour of the Lagrangian Multipler Parameter, a 2-entry vector

7.0 Comments and Conclusion

Table 1 exhibits the number of cycles, objective function values and constraint satisfaction with sampled numerical solutions appreciating to 1.3953. We observe that as the penalty parameter gets bigger, the objective function value increases and appreciates to 1.3953 at the 6^{th} and 7^{th} iterations. Also, the constraint satisfaction approaches zero initially from 1.6530E-1 to 5.0297E-7. This signifies that the computed numerical solution 1.3953 is the approximate solution to the original problem with a level of tolerance (-.0025, +.0025) used in the programming.

Table 2 shows the performance of the developed Algorithm by examining the convergence analysis. The state variables generated by conjugate gradient method (CGM) are analyzed and found with ratios between 0 and 1. This shows that the analysis is linearly convergent.

Figure 1(a, b, c, and d) shows graphically the behaviour of the state and control variables.. Of interest is the fact that the control variable approaches zero, thus signifying control compliance at the end of the iteration.

Figure 2 (a and b) shows the sinusoidal behavior of the Lagrangian multiplier parameter. The first entry with domain [0,1], behaves sinusoidally starting with amplitude between -2.00 and +.200 and ending with amplitude between -.700 and +.100. As for the second entry, it behaves sinusoidally also, starting with amplitude between -.605 and -.65 and ending approximately on zero. This shows the deterioration of the Lagrangian Multiplier to appreciate the objective function value 1.3953.

The Algorithm shows its effectiveness as it gives approximate solution to problem not solvable analytically, since the control variable appreciates to zero as iteration continues. Therefore, the developed Algorithm has demonstrated its significance, effectiveness and reliability. So problems belonging to this type of research not solvable analytically can be solved approximately by this developed Algorithm.

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