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Uniform Order Zero- Stable K-Step Block Methods For Initial Value Problems Of Ordinary Differential Equations

A. M. Badmus* and D. W. Mshelia
***Department of Mathematics and Computer Science,**
Nigerian Defence Academy Kaduna, Nigeria.
Department of Mathematics and Statistic Umar Ibn Ibrahim Elkanami.
College of Education, science and Technology, Bama Borno State.

Abstract

Efficient and accurate uniform order hybrid block methods are proposed in this paper. Specifically the schemes are of uniform order 4 for k_i $(i = 2, 3)$. The block schemes are adequate, and accurate and also yield good stability properties. The traditional implementation of single step method is avoided by new proposed method over block steps, which requires no starting value. Moreover the speed of integration is enhanced for k_i $(i = 2, 3)$ by $2k$ and $(2k - 1)$ successive steps movement at a time (block steps). Numerical examples are given to illustrate the performance of the new procedures.

Keywords: Zero stability, Hybrid, k –step, Block methods, first order initial value problem

1.0 Introduction

Linear multi-step methods constitute a powerful class of numerical procedures for solving initial value problems of ordinary differential equations. This approach requires starting value and it used in predictor-corrector mode of implementation.(see [1]). Since the current trend in numerical solution of ordinary differential equations is toward accurate, efficient and excellent algorithms to handle it effectively. There is need to develop a high order schemes to address these problems.

In this paper, we extended the methods in [2] in proposing some hybrid methods of 4-point block method at $k = 2$ and also 5-point block method at $k = 3$ which are all of Uniform order 4 through multi-step collocation approach. We are able to obtain a continuous formulation which evaluated at some grid and off grid points of $x = x_{n+j}, j \epsilon [0,3]$ provided enough discrete schemes to form a block method approach for various values of k.

2.0 Construction of the method

We consider a collocation polynomial of the form

$$y(x) = \sum_{j=0}^{k-1} \emptyset_j(x) y_{n+j} + h \sum_{j=0}^{m-1} \varphi_j(x) f(\bar{x}_{n+j}, y(\bar{x}_{n+j})) \qquad (2.1)$$

Where $\emptyset_j(x)$ and $\varphi_j(x)$ are the polynomial of the form.

$$\emptyset_j(x) = \sum_{j=0}^{m+t-1} \varphi_j x^j \qquad (2.2a)$$

$$\emptyset_j(x) = j \sum_{j=1}^{m+t-1} h\, \varphi_j x^{j-1} \qquad (2.2b)$$

where m and t are distinct collocation and interpolation points in the method. The method (2.1) satisfies the following conditions.

$$y(x_{n+j}) = y_{n+j}\; j \in (0,1,k-1) \qquad (2.3)$$

$$y(\bar{x}_j) = f\left(\bar{x}, \bar{y}(x_j)\right) \;\; j \in (0,1,..m-1) \qquad (2.4)$$

Where $\bar{x}_j, y(\bar{x}_j)$ are collocation points used. The following conditions are posed on $\emptyset_j$ and $\varphi_j(x)$.

$$\emptyset_j(x_{n+j}) = \delta_{ij} \qquad j = 0,1,..k-1$$

$$h\varphi_j(x_{n+j}) = 0 \qquad j = 0,1, \ldots m-1$$

Corresponding author: ***A. M. Badmus.*** E-mail: ademolabadmus@gmail.com, Tel. +2348028412497.

$$\emptyset'_j(x_{n+j}) = 0$$
$$h\varphi'_j(x_{n+j}) = \delta_{ij} \qquad (2.5)$$

Arranging (2.2a) and (2.2b) in a matrix equation form DC=I. where I is the identity matrix , we have

$$D=\begin{bmatrix} 1 & x_n & x_n^2 & . & . & x_n^{t+m-1} \\ 1 & x_n & x_n^2 & . & . & x_n^{t+m-1} \\ 0 & 1 & 2x_{n+k-1} & . & . & (t+m-1)x_{n+k-1}^{t+m-1} \\ 0 & 1 & 2x_{n+k+1} & . & . & (t+m-1)\,x_{n+k+1}^{t+m-1} \\ . & . & . & . & . & . \\ 0 & 1 & 2x_{n+k-1} & . & . & (t+m-1)x_{n+k-1}^{t+m-1} \\ 0 & 1 & 2x_{n+k-1} & . & . & (t+m-1)\,x_{n+k-1}^{t+m-1} \end{bmatrix} \qquad (2.6)$$

where matrix D in 2.6 is of dimension $(m+t)$ x $(m+t)$, and

$$C=\begin{bmatrix} \emptyset_{01} & . & . & \emptyset_{t-1,1} & h\emptyset_{0,1} & . & . & . & h\varphi_{s-1,1} \\ \emptyset_{02} & . & . & \emptyset_{t-1,2} & h\emptyset_{0,2} & . & . & . & h\varphi_{s-1,2} \\ \emptyset_{03} & . & . & \emptyset_{t-1,3} & h\emptyset_{0,3} & . & . & . & h\varphi_{n+1,1} \\ \emptyset_{04} & . & . & \emptyset_{t-1,4} & h\emptyset_{0,4} & . & . & . & h\varphi_{s-1,4} \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ \emptyset_{0,t+m} & . & . & \emptyset_{t-1,t+m} & h\emptyset_{0,t+m} & . & . & . & h\varphi_{s-1,t+m} \end{bmatrix} \qquad (2.7)$$

also of dimension $(m+t)$ x$(m+t)$

Columns of C which give the continuous coefficients $\emptyset_j(x)$ and $\varphi_j(x)$ $(j = 0,1,2,..m-1)$ can be obtained from the corresponding columns of D^{-1}.

Thus we can express equation (2.1) explicitly in the form

$$y(x) = (y_n, y_{n+u}, y_{n+w}, y_{n+v}, .. f_n, f_{n+w}, f_{n+v})C^T(1, x, x^2, \dots, x^m)^T \qquad (2.8)$$

where superscript T denotes "transpose of" the matrix and the vector $(1, x, x^2, \dots, x^m)$ in (2.8).

Definitions 1.0

a) A linear Multi-step method (2.1) is said to be Zero-stable if no root of the first characteristic polynomial $\rho(x) = \sum_{j=0}^{k} \alpha_j r^j$ has modulus greater than one and if every root with modulus one is simple.

b) A linear Multi-step method (2.1) is said to be A- stable if its region of absolute stability contains the whole of the left hand half plane $Re\ (h\lambda) < 0$

3.0 Specification of the Method

In this section, we consider two hybrid block methods of uniform order 4 for $k = 2$ and 3 respectively.

For $k = 2$, consider an approximate solution to (2.1) in the form

$$y_k(x) = \sum_{j=0}^{m+t-1} a_j x^j \qquad (3.1)$$

$$y_k(x) = \sum_{j=1}^{m+t-1} j\, a_j x^{j-1} = f(x, y) \qquad (3.2)$$

where a_j are the parameters to be determined.

Specification in this method $m = 3$ and $t = 3$, we interpolated (3.1) and collocate (3.2) at $x_{n+j}, j = u, w, v$ to get the following system of non linear equations.

$$a_0 + a_1x_{n+u} + a_2x_{n+u}^2 + a_3x_{n+u}^3 + a_4x_{n+u}^4 + a_5x_{n+u}^5 = y_{n+u}$$
$$a_0 + a_1x_{n+w} + a_2x_{n+w}^2 + a_3x_{n+w}^3 + a_4x_{n+w}^4 + a_5x_{n+w}^5 = y_{n+w}$$
$$a_0 + a_1x_{n+v} + a_2x_{n+v}^2 + a_3x_{n+v}^3 + a_4x_{n+v}^4 + a_5x_{n+v}^5 = y_{n+v}$$
$$a_1 + 2a_2x_{n+u} + 3a_3x_{n+u}^2 + 4a_4x_{n+u}^3 + 5a_5x_{n+u}^4 = f_{n+u}$$
$$a_1 + 2a_2x_{n+w} + 3a_3x_{n+w}^2 + 4a_4x_{n+w}^3 + 5a_5x_{n+w}^4 = f_{n+w}$$
$$a_1 + 2a_2x_{n+v} + 3a_3x_{n+v}^2 + 4a_4x_{n+v}^3 + 5a_5x_{n+v}^4 = f_{n+v}$$
$$(3.3)$$

When (3.3) is arranged in matrix equation form we have

$$\begin{bmatrix} 1 & x_{n+u} & x_{n+u}^2 & x_{n+u}^3 & x_{n+u}^4 & x_{n+u}^5 \\ 1 & x_{n+w} & x_{n+w}^2 & x_{n+w}^3 & x_{n+w}^4 & x_{n+w}^5 \\ 1 & x_{n+v} & x_{n+v}^2 & x_{n+v}^3 & x_{n+v}^4 & x_{n+v}^5 \\ 0 & 1 & 2x_{n+u} & 3x_{n+u}^2 & 4x_{n+u}^3 & 5x_{n+u}^4 \\ 0 & 1 & 2x_{n+w} & 3x_{n+w}^2 & 4x_{n+w}^3 & 5x_{n+w}^4 \\ 0 & 1 & 2x_{n+v} & 3x_{n+v}^2 & 4x_{n+v}^3 & 5x_{n+v}^4 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} = \begin{bmatrix} y_{n+u} \\ y_{n+w} \\ y_{n+v} \\ f_{n+u} \\ f_{n+w} \\ f_{n+v} \end{bmatrix} \qquad (3.4)$$

Our proposed continuous formula is of the form

$$y(x) = \propto_0 (x)y_{n+u} + \propto_1 (x)y_{n+w} + \propto_2 (x)y_{n+v} + h\{\beta_0(x)f_{n+u} + \beta_1(x)f_{n+w} + \beta_2(x)f_{n+v}\} \qquad (3.5)$$

Using Mathematical soft ware (Maple) to perform some algebraic manipulation to determine values of a_j ($j = 0,1,2, \dots k+3$) we obtain the continuous formulation of the form.

$$y(x) = \{\frac{1}{h^5(wv-uv-wu+u^2)(w-u)^2(v^2-2uv+u^2)}(-3v^3h^5uw^2 + 5u^2h^5w^2v^2 - 3v^2h^5w^3u + v^3h^5w^3)$$
$$+\frac{1}{(wu-uv-wu+u^2)(v-u)^2h^5(-2wu+u^2+w^2)}(2(-5wh^4v^2u^2 - 5w^2h^4vu^2 + 3w^3h^4vu +$$
$$4w^2h^4v^2u + 3wh^4v^3u))(x-x_n) - \frac{1}{(v^2-2uv+u^2)(w-u)h^5(-2wu+u^2+w^2)(v-u)}((7v^2h^3wu + 3w^3h^3w +$$
$$4v^2h^3w^2 + 3vw^3h^3 + 3v^3h^3u - 5v^2u^2h^3 + 3w^3h^3u - 5w^2h^3u^2 - 20vwu^2h^3 + 7vw^2h^3u)(x-$$
$$x_n)^2) + \frac{1}{(wu-uv-wu+u^2)(v-u)^2h^5(-2wu+u^2+w^2)}(2(\ h^2w^3 + 4w^2vh^2 + w^2uh^2 - 2wuvh^2 +$$
$$4v^2wh^2 - 5u^2wh^2 - 5u^2vh^2 + v^3h^2 + v^2h^2u)(x-x_n)^3)$$
$$-\frac{(4v^2h+7vhw-5uvh+4w^2h-5u^2h-5uwh)}{(v^2-2uv+u^2)(w-u)h^5(-2wu+u^2+w^2)(v-u)}(x-x_n)^4 + \frac{2(v-2u+w)(x-x_n)^5}{(v^2-2uv+u^2)(w-u)h^5(-2wu+u^2+w^2)(v-u)})\}y_{n+u}$$
$$+\{\frac{uh(3wh^4v^2u^2-5w^2h^4v^2u+3wh^4v^3u-h^4v^3v^2)}{((w-u)(-2uw+u^2+w^2)(-2wv+v^2+w^2)(-w+v)h^5)}$$
$$-\frac{2(4wh^4v^2u^2-5w^2h^4vu^2-5w^2h^4v^2u+3wh^4v^3u+3vh^4w)(x-x_n)}{(-2vwu+vu^2+vw^2-w^3+2uw^2-u^2w)(w-u)(-w+v)^2h^5} + \ (7v^2h^3wu + 3v^3h^3w - 5v^2h^3w^2 +$$
$$3v^3h^3u + 4v^2u^2h^3 - 5w^2h^3u^2 + 3u^3h^3v + 3wu^3h^3 + 7vwu^2h^3 - 20vw^2h^3u)(x-$$
$$x_n)^2)/((w-u)(-2wu+u^2+w^2)(-2wv+v^2+w^2)(-w+v)h^5)$$
$$-\frac{(2(v^3h^2+wv^2h^2+4v^2h^2u-5w^2vh^2-2wuvh^2+4u^2vh^2+u^2h^2w+h^2u^3-5h^2w^2u)(x-x_n)^3)}{(w-u)(-2wu+u^2+w^2)(-2wv+v^2+w^2)(-w+v)h^5}$$
$$+(\frac{-5w^2h-5vhw-5uhw+7uvh+4v^2h+4u^2h)(x-x_n)^4}{(-2vwu+vu^2+vw^2-w^3+2uw^2-u^2w)(w-u)(-w+u)^2h^5} - \frac{2(v-2w+u)(x-x_n)^5}{(w-u)(-2wu+u^2+w^2)(-2wv+v^2+w^2)(-w+v)h^5}\}y_{n+w}$$
$$+\{\frac{uh(-3w^2h^4vu^2-3w^3h^4vu+5w^2h^4v^2u+h^4w^3u^2)}{(v^3-v^2u-2v^2w+2vwu+vw^2-uw^2)(v-u)(-wv+wu+v^2-uv)h^5}$$
$$-\frac{(2(5wh^4v^2u^2-4w^2h^4vu^2-3w^3h^4vu-3vh^4wu^3)(x-x_n)}{(-w+v)(-2v^3w-2v^3u+v^4+w^2v^2+w^2u^2+u^2v^2-2w^2uv+4uwv^2-2u^2wv)h^5(v-u)}$$
$$+\frac{1}{(-wv+wu+v^2-uv)^3h^5}(20v^2h^3wu + 5v^2h^3w^2 - 3vw^3h^3 + 5v^2h^3u^2 - 3uh^3w^3 - 4u^2h^3w^2 -$$
$$3vh^3u^3 - 3u^3h^3w - 7vwu^2h^3 - 7vuh^3w^2)(x-x_n)^2$$
$$-\frac{2(-h^2w^2-4h^2w^2u-h^2vw^2+2wuvh^2+5wh^2v^2-4h^2u^2w-vh^2u^2-h^2u^3+5v^2h^2u)(x-x_n)^3}{(v-u)^2(v^3-v^2u-2v^2w+2vwu+vw^2-uw^2)(-w+v)h^5}$$
$$+\frac{(-4w^2h-7uhw+5vhw+5uvh-4u^2h+5v^2h)(x-x_n)^4}{(-w+v)(-2v^3w-2v^3u+v^4+w^2v^2+u^2v^2-2w^2uv+4uwv^2-2u^2wv)h^5(v-u)}$$
$$-\frac{2(-w+2v-u)(x-x_n)^5}{h^5(-wv+wu+v^2-uv)(-2v^3w-2v^3u+v^4+w^2v^2+u^2v^2-2w^2uv+4uwv^2-2u^2wv)}\}y_{n+v}$$

$$+\{\frac{v^2h^4w^2(uh)}{(-2wu+u^2+w^2)(v^2-2uv+u^2)h^4}+\frac{(v^2h^4w^2+2v^2wh^4u+2w^2uh^4v)(x-x_n)}{h^4(wv-uv-wu+u^2)(v-u)(w-u)}$$
$$-\frac{(v^2h^3u+2v^2wh^3+4vuh^3w+2vw^2h^3+uw^2h^3)(x-x_n)^2}{(v-u)^2(-2wu+u^2+w^2)h^4}$$

$$+\frac{(v^2h^2+4vh^2w+2uvh^2+w^2h^2+2h^2wu)(x-x_n)^3}{(wv-vw-wu+u^2)(v-u)(w-u)h^4}-\frac{(2vh+uh+2wh)(x-x_n)^4}{h^4(-2wv+u^2+w^2)v^2-2uv+u^2)}$$
$$+\frac{(x-x_n)^5}{(v^2-2uv+u^2)(-2wu+u^2+w^2)h^4}\}f_{n+u}+\{-\frac{(v^2wh^4u)(uh)}{(w-u)^2(-2wu+v^2+w^2)h^4}$$
$$+\frac{(2v^2wh^4u+h^4v^2u^2+2vwh^4u^2)(x-x_n)}{h^4(-2vwu+vu^2+vw^2-w^3+2uw^2-u^2w)(-w+v)}-\frac{(v^2h^3v+2v^2h^3u+4vh^3wu+2vh^3u^2+h^3wu^2)(x-x_n)^2}{(-w+v)^2(-2wu+u^2+w^2)h^4}$$
$$+\frac{(v^2h^2+2vh^2w+4uvh^2+u^2h^2+2h^2wu)(x-x_n)^3}{(w-u)^2(-2wv+v^2+w^2)h^4}+\frac{(wh+2vh+2uh)(x-x_n)^4}{(-2vwu+vu^2+vw^2-w^3+2uw^2-u^2w)(-w+v)h^4}$$
$$+\frac{(x-x_n)^5}{(-2vw+v^2+w^2(-2wu+u^2+w^2)h^4}\}f_{n+w}$$
$$+\{\frac{-(vw^2h^4u)(uh)}{(-2v^3w-2v^3u+v^4+w^2v^2+u^2w^2+u^2v^2-2w^2uv+4uwv^2-2u^2wv)h^4}$$
$$+\frac{(w^2h^4u^2+2vh^4w^2u+2vwh^4u^2)(x-x_n)}{(-2v^3w-2v^3u+v^4+w^2v^2+u^2w^2+u^2v^2-2w^2uv+4uwv^2-2u^2wv)h^4}$$
$$-\frac{(vh^3u^2+vw^2h^3+vw^2h^3+4vh^3wu+2h^3wu^2+2h^3w^2u)(x-x_n)^2}{(-wv+wu+v^2uv)^2h^4}$$
$$+\frac{(h^2w^2+2vwh^2+4h^2wu+h^2u^2+2h^2uv)(x-x_n)^3}{(v-u)(v^3-v^2u-2v^2w+2vwu+vw^2-uw^2)h^4}-\frac{(2wh+vh+2uh)(x-x_n)^4}{(-w+v)(-v^2w-2v^2u-u^2w+2vwu+v^3+vu^2)h^4}$$
$$+\frac{(x-x_n)^5}{(-2v^3w-2v^3u+v^4+w^2v^2+u^2w^2+u^2v^2-2w^2uv+4uwv^2-2u^2wv)h^4}\}f_{n+v} \qquad (3.6)$$

Choose $u=\frac{1}{2}$, $w=1$, $v=\frac{3}{2}$ in equation (3.6) and evaluate it at $x=x_n$ *and* $x=x_{n+2}$. Also evaluate the first derivative of (3.6) at $x=x_n$ *and* $x=x_{n+2}$ to obtain the following discrete schemes.

$$y_n+18y_{n+\frac{1}{2}}-9y_{n+1}-10y_{n+\frac{3}{2}}=\frac{h}{2}[-9f_{n+\frac{1}{2}}-18f_{n+1}-3f_{n+\frac{3}{2}}\ \ ]$$
$$y_{n+2}-10y_{n+\frac{1}{2}}-9y_{n+1}+18y_{n+\frac{3}{2}}=\frac{h}{2}[\,3f_{n+\frac{1}{2}}+18f_{n+1}+9f_{n+\frac{3}{2}}\,]$$
$$66y_{n+\frac{3}{2}}+48y_{n+1}-114y_{n+\frac{1}{2}}=h[-f_n+24f_{n+\frac{1}{2}}+57f_{n+1}+10f_{n+\frac{3}{2}}]$$
$$114y_{n+\frac{3}{2}}-48y_{n+1}-66y_{n+\frac{1}{2}}=h\left[10f_{n+\frac{1}{2}}+57f_{n+1}+24f_{n+\frac{3}{2}}-f_{n+2}\right] \qquad (3.7)$$

Equation (3.7) is of uniform order $[4,4,4,4]^T$ and Error constants of $[-\frac{45}{64},\frac{85}{64},\frac{291}{64},\frac{435}{64}]^T$

For $k=3$ Specifying in this method $m=3$ and $t=4$, we interpolated (3.1) at x_{n+j}, $(j=0,\frac{1}{2},1,\frac{3}{2})$ and collocate (3.2) at x_{n+j}, $j=1,2,3$ to get the following system of non linear equations

.
$$a_0+a_1x_n+a_2x_n^2+a_3x_n^3+a_4x_n^4+a_5x_n^5+a_6x_n^6=y_n$$
$$a_0+a_1x_{n+\frac{1}{2}}+a_2x^2_{n+\frac{1}{2}}+a_3x^3_{n+\frac{1}{2}}+a_4x^4_{n+\frac{1}{2}}+a_5x^5_{n+\frac{1}{2}}+a_6x^6_{n+\frac{1}{2}}=y_{n+\frac{1}{2}}$$
$$a_0+a_1x_{n+1}+a_2x_{n+1}^2+a_3x_{n+1}^3+a_4x_{n+1}^4+a_5x_{n+1}^5+a_6x_{n+1}^6=y_{n+1}$$
$$a_0+a_1x_{n+\frac{3}{2}}+a_2x^2_{n+\frac{3}{2}}+a_3x^3_{n+\frac{3}{2}}+a_4x^4_{n+\frac{3}{2}}+a_5x^5_{n+\frac{3}{2}}+a_6x^6_{n+\frac{3}{2}}=y_{n+\frac{3}{2}}$$
$$a_1+2a_2x_{n+1}+3a_3x_{n+1}^2+4a_4x_{n+1}^3+5a_5x_{n+1}^4+6x_{n+1}^5=f_{n+1}$$
$$a_1+2a_2x_{n+2}+3a_3x_{n+2}^2+4a_4x_{n+2}^3+5a_5x_{n+2}^4+6x_{n+2}^5=f_{n+2}$$
$$a_1+2a_2x_{n+3}+3a_3x_{n+3}^2+4a_4x_{n+3}^3+5a_5x_{n+3}^4+6x_{n+3}^5=f_{n+3} \qquad (3.8)$$

Our proposed continuous formula is of the form

$$y(x) = \alpha_0\,(x) y_n + \alpha_1\,(x) y_{n+\frac{1}{2}} + \alpha_2\,(x) y_{n+1} + \alpha_3\,(x) y_{n+\frac{3}{2}} + h\{\beta_0(x) f_{n+1} + \beta_1(x)\, f_{n+2} + \beta_2(x) f_{n+3}\} \tag{3.9}$$

Using Mathematical soft ware (Maple) to perform some algebraic manipulation to determine values of a_j $(j = 0,1,2, \ldots k+4)$ we obtain the continuous formulation of the form.

$$\begin{aligned}
y(x) = &\Big[\frac{1}{9}\,\frac{(9h^6)}{h^6} - \frac{2}{9}\frac{(33h^5)(x-x_n)}{h^6} + \frac{1}{9}\frac{(193h^4)(x-x_n)^2}{h^6} - \frac{32}{9}\,\frac{(9h^3)(x-x_n)^3}{h^6} + \frac{8}{9}\,\frac{(29h^2)(x-x_n)^4}{h^6} \\
&\quad - \frac{32}{3}\,\frac{(h)(x-x_n)^5}{h^6} + \frac{16}{9}\,\frac{(x-x_n)^6}{h^6}\Big] y_n \\
&+ \Big[-\frac{6(3h^5)(x-x_n)}{h^6} + \frac{6(22h^4)(x-x_n)^2}{h^6} - \frac{2(157h^3)(x-x_n)^3}{h^6} + \frac{24(14h^2)(x-x_n)^4}{h^6} \\
&\quad - \frac{24(7h)(x-x_n)^5}{h^6} + \frac{32(x-x_n)^6}{h^6}\Big] y_{n+\frac{1}{2}} \\
&+ \Big[\frac{6(3h^5)(x-x_n)}{h^6} - \frac{3(35h^4)(x-x_n)^2}{h^6} + \frac{32(7h^3)(x-x_n)^3}{h^6} - \frac{24(9h^2)(x-x_n)^4}{h^6} + \frac{96h(x-x_n)^5}{h^6} \\
&\quad - \frac{16(x-x_n)^6}{h^6}\Big] y_{n+1} \\
&+ \Big[\frac{2}{9}\,\frac{(33h^5)(x-x_n)}{h^6} - \frac{2}{9}\,\frac{(218h^4)(x-x_n)^2}{h^6} + \frac{2}{9}\,\frac{(549h^3)(x-x_n)^3}{h^6} + \frac{8}{9}\,\frac{(164h^2)(x-x_n)^4}{h^6} \\
&\quad + \frac{8}{3}\,\frac{(31h)(x-x_n)^5}{h^6} - \frac{160}{9}\,\frac{(x-x_n)^6}{h^6}\Big] y_{n+\frac{3}{2}} \\
&+ \Big[-\frac{(9h^5)(x-x_n)}{h^5} + \frac{(48h^4)(x-x_n)^2}{h^5} - \frac{(97h^3)(x-x_n)^3}{h^5} + \frac{2(47h^2)(x-x_n)^4}{h^5} - \frac{4(11h)(x-x_n)^5}{h^5} \\
&\quad + \frac{8(x-x_n)^6}{h^5}\Big] f_{n+\frac{1}{2}} \\
&+ \Big[-\frac{(9h^5)(x-x_n)}{h^5} + \frac{(57h^4)(x-x_n)^2}{h^5} - \frac{8(17h^3)(x-x_n)^3}{h^5} + \frac{8(19h^2)(x-x_n)^4}{h^5} - \frac{16(5h)(x-x_n)^5}{h^5} \\
&\quad + \frac{16(x-x_n)^6}{h^5}\Big] f_{n+1} \\
&+ \Big[-\frac{1}{3}\,\frac{(3h^5)(x-x_n)}{h^5} + \frac{1}{3}\,\frac{(20h^4)(x-x_n)^2}{h^5} - \frac{1}{3}\,\frac{(51h^3)(x-x_n)^3}{h^5} + \frac{2}{3}\,\frac{(31h^2)(x-x_n)^4}{h^5} \\
&\quad - \frac{4(3h)(x-x_n)^5}{h^5} + \frac{8}{3}\,\frac{(x-x_n)^6}{h^5}\Big] f_{n+\frac{3}{2}}
\end{aligned} \tag{3.10}$$

Evaluating (3.10) at $x = x_{n+j},\ j = 2,3$ and its first derivative at $x = x_{n+j},\ j = 0,2,3$ yields the following

$$y_{n+2} - 28y_{n+\frac{1}{2}} + 28y_{n+\frac{3}{2}} - y_n = h[6f_{n+\frac{1}{2}} + 18f_{n+1} + 6f_{n+\frac{3}{2}}]$$

$$y_{n+3} - 100y_n - 2376y_{n+\frac{1}{2}} + 675y_{n+1} + 1800y_{n+\frac{3}{2}} = h[540f_{n+\frac{1}{2}} + 1350f_{n+1} + 300f_{n+\frac{3}{2}}]$$

$$22y_{n+\frac{3}{2}} + 54y_{n+1} - 54y_{n+\frac{1}{2}} - 22y_n = h[3f_n + 27f_{n+\frac{1}{2}} + 27f_{n+1} + 3f_{n+\frac{3}{2}}]$$
$$562y_{n+\frac{3}{2}} + 54y_{n+1} - 594y_{n+\frac{1}{2}} - 22y_n = h\left[129f_{n+\frac{1}{2}} + 369f_{n+1} + 105f_{n+\frac{3}{2}} - 3f_{n+2}\right]$$
$$15520y_{n+\frac{3}{2}} + 7020y_{n+1} - 21600y_{n+\frac{1}{2}} - 940y_n = h\left[4968f_{n+\frac{1}{2}} + 12015f_{n+1} + 2520f_{n+\frac{3}{2}} - 3f_{n+3}\right]$$

(3.11)

Equation (3.11) is of uniform order $[4,4,4,4,4]^T$ and Error constants of $[\frac{1935}{16}, \frac{65}{32}, \frac{117}{64}, \frac{2295}{64}, 1035]^T$

4.0 Block Stability analysis

Following [3] and [4], we shall normalize each of the proposed block method. We multiply the Matrix $A^{(0)}, A^{(1)}, B^{(0)}$ and $B^{(1)}$ in the block with the inverse of $A^{(0)}$

For the Block scheme of $k = 2$, we obtain the following results.

When equation (3.7) is put in matrix-form we have,

$$\begin{pmatrix} 18 & -9 & -10 & 0 \\ -10 & -9 & 18 & 1 \\ -114 & 48 & 66 & 0 \\ -66 & -48 & 114 & 0 \end{pmatrix} \begin{pmatrix} y_{n+\frac{1}{2}} \\ y_{n+1} \\ y_{n+\frac{3}{2}} \\ y_{n+2} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} y_{n-\frac{3}{2}} \\ y_{n-1} \\ y_{n-\frac{1}{2}} \\ y_n \end{pmatrix} +$$

$$+ \begin{pmatrix} -\frac{9}{2} & -\frac{18}{2} & -\frac{3}{2} & 0 \\ \frac{3}{2} & \frac{18}{2} & \frac{9}{2} & 0 \\ 24 & 57 & 10 & 0 \\ 10 & 57 & 24 & -1 \end{pmatrix} \begin{pmatrix} f_{n+\frac{1}{2}} \\ f_{n+1} \\ f_{n+\frac{3}{2}} \\ f_{n+2} \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} f_{n-\frac{3}{2}} \\ f_{n-1} \\ f_{n-\frac{1}{2}} \\ f_n \end{pmatrix}$$

(4.1)

Let

$$\boldsymbol{A}^{(0)} = \begin{pmatrix} 18 & -9 & -10 & 0 \\ -10 & -9 & 18 & 1 \\ -114 & 48 & 66 & 0 \\ -66 & -48 & 114 & 0 \end{pmatrix} \qquad \boldsymbol{A}^{(1)} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\boldsymbol{B}^{(0)} = \begin{pmatrix} -\frac{9}{2} & -\frac{18}{2} & -\frac{3}{2} & 0 \\ \frac{3}{2} & \frac{18}{2} & \frac{9}{2} & 0 \\ 24 & 57 & 10 & 0 \\ 10 & 57 & 24 & -1 \end{pmatrix} \qquad \boldsymbol{B}^{(1)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

We shall normalize the block method (4.1) by multiply matrices $A^{(0)}, A^{(1)}, B^{(0)}$and $B^{(1)}$ in the block with inverse of $A^{(0)}$.ie

$$(\boldsymbol{A}^{(\mathbf{0})})^{-\mathbf{1}} = \begin{pmatrix} -1 & 0 & -\frac{251}{1440} & \frac{19}{1440} \\ -1 & 0 & -\frac{29}{180} & \frac{1}{180} \\ -1 & 0 & -\frac{27}{160} & \frac{3}{160} \\ -1 & 1 & -\frac{7}{45} & -\frac{7}{45} \end{pmatrix}$$

we thus obtain the normalized form for (4.1) as

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_{n+\frac{1}{2}} \\ y_{n+1} \\ y_{n+\frac{3}{2}} \\ y_{n+2} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_{n-\frac{3}{2}} \\ y_{n-1} \\ y_{n-\frac{1}{2}} \\ y_n \end{pmatrix} + \begin{pmatrix} \frac{323}{720} & -\frac{11}{60} & \frac{227}{180} & 0 \\ \frac{31}{45} & \frac{2}{15} & \frac{47}{90} & 0 \\ \frac{51}{80} & \frac{9}{20} & \frac{39}{20} & 0 \\ \frac{32}{45} & \frac{4}{15} & -\frac{598}{45} & 0 \end{pmatrix} \begin{pmatrix} f_{n+\frac{1}{2}} \\ f_{n+1} \\ f_{n+\frac{3}{2}} \\ f_{n+2} \end{pmatrix} +$$

$$\begin{pmatrix} 0 & 0 & 0 & \frac{251}{1440} \\ 0 & 0 & 0 & \frac{29}{180} \\ 0 & 0 & 0 & \frac{27}{160} \\ 0 & 0 & 0 & \frac{7}{45} \end{pmatrix} \begin{pmatrix} f_{n-\frac{3}{2}} \\ f_{n-1} \\ f_{n-\frac{1}{2}} \\ f_n \end{pmatrix} \qquad (4.2)$$

The first characteristic polynomial of the block (4.2) is

$$\rho(R) = det\left[RA^{(0)} - A^{(1)}\right] = det\left[R\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right] = 0$$

$$= det\left[\begin{pmatrix} R & 0 & 0 & -1 \\ 0 & R & 0 & -1 \\ 0 & 0 & R & -1 \\ 0 & 0 & 0 & R-1 \end{pmatrix}\right] = R^4 - R^3 = 0 \qquad (4.3)$$

which implies that $R_1 = R_2 = R_3 = 0$ and $R_{4=}1$. From definition (1.0a) and the equation (4.3), the block method (3.7) is zero-stable and also consistent as its order P= $[4, 4, 4,4]^T > 1$, thus it is convergent.

Remark:

When performing the same analysis and using similar computation for $k = 3$, we have the following results.

$$\left(det\begin{bmatrix} R & 0 & 0 & 0 & 0 \\ 0 & R & 0 & 0 & 0 \\ 0 & 0 & R & 0 & 0 \\ 0 & 0 & 0 & R & 0 \\ 0 & 0 & 0 & 0 & R \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \right) = 0$$

$$det\begin{bmatrix} R & 0 & 0 & 0 & -1 \\ 0 & R & 0 & 0 & -1 \\ 0 & 0 & R & 0 & -1 \\ 0 & 0 & 0 & R & -1 \\ 0 & 0 & 0 & 0 & R-1 \end{bmatrix} = 0$$

$$R^5 - R^4 = 0 \qquad (4.4)$$

which implies that $R_1 = R_2 = R_3 = R_4 = 0$ and $R_{5=}1$. From definition (1.0a) and the equation (4.4), the block method (3.11) is zero-stable and also consistent as its order P= $[4,4,4,4,4]^T > 1$, thus it is convergent.

5.0 Numerical experiments with the new block multistep methods.

Example 5.1

$xy' + 2y = 4x^2 \qquad y(1) = 2$

Exact solution is $y(x) = \frac{x^2+1}{x^2}$

Table 1: Numerical Results and Absolute error of example 5.1

x	Theoretical solution	Block Method k=2	Block Method k-3	Error at k=2	Error at k=3
1.1	2.036446280992	2.036445898000	2.036446163000	3.82992E-07	1.17992E-07
1.2	2.134444444444	2.134444478000	2.13444396000	3.3556E-08	4.84444E-07
1.3	2.281715976331	2.281715923000	2.281712386000	5.3331E-08	3.59033E-06
1.4	2.470204081633	2.470204147000	2.470200978000	6.5367E-08	3.10363E-06
1.5	2.694444444444	2.694444443000	2.694441713000	1.444E-09	2.73144E-06
1.6	2.950625000000	2.950625033000	2.950624169000	3.3E-08	8.31E-07

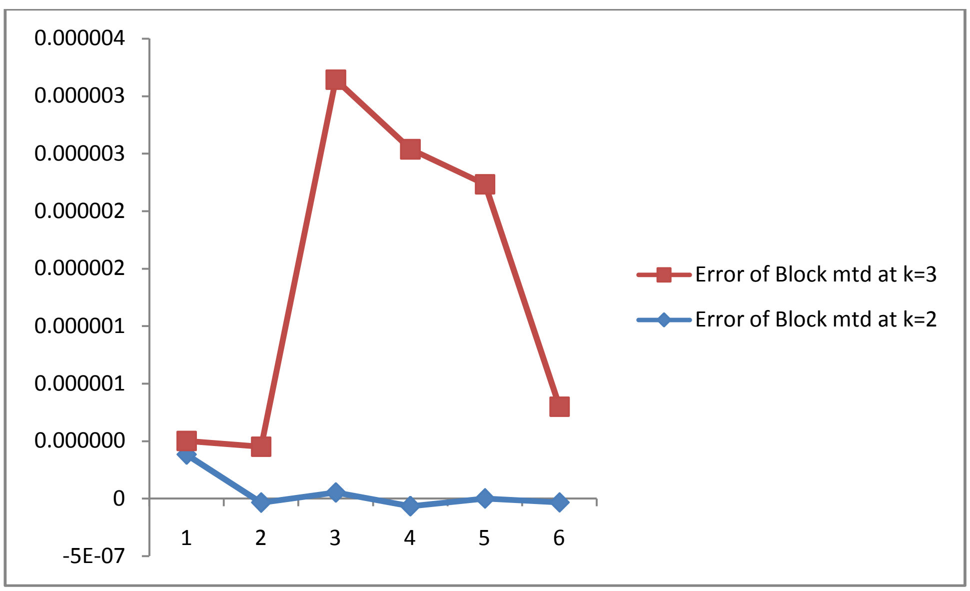

Figure1: error graph of example 5.1

Example 5.2

$$y' = \frac{3x^2 + 4x + 2}{2(y-1)} \qquad y(0) = -1$$

Exact solution is $y(x) = 1 - \sqrt{4 + x^3 + 2x^2 + 2x}$

Table 2: Numerical Results and Absolute error of example 5.2

x	Theoretical sol	Block Mtd k=2	Block Mtd k-3	Error at k=2	Error at k=3
1.1	-1.054507240192	-1.054507238156	-1.054507240148	2.036E-09	4.39999E-11
1.2	-1.118490028298	-1.118490028442	-1.118490028288	1.44E-10	1E-11
1.3	-1.192487172140	-1.192487170841	-1.192487169641	1.299E-09	2.499E-09
1.4	-1.276839915321	-1.276839915367	-1.276839912599	4.60001E-11	2.722E-09
1.5	-1.371708245126	-1.371708244915	-1.371708242602	2.11E-10	2.524E-09
1.6	-1.477095072863	-1.477095072719	-1.477095063727	1.44E-10	9.136E-09

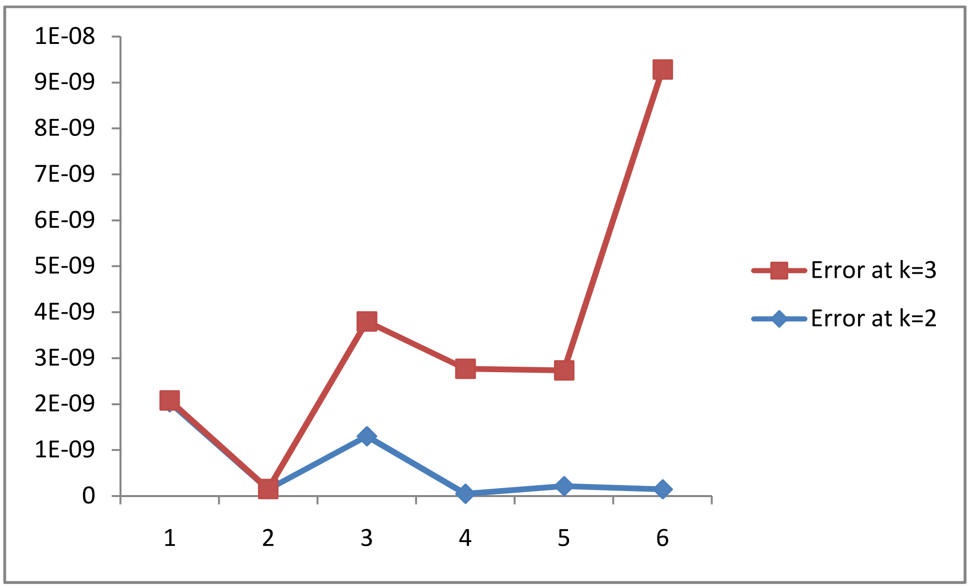

Figure 2: error graph of example 5.2

5.0 Discussion of Results

The two problems were tested with proposed schemes derived from $k_i(i = 2,3)$. We observed that the proposed block scheme from $k = 2$ performed better in the two problems. (See Table 1 and 2 also figures (1 and 2).

6.0 Conclusion

We want to draw the following conclusions with these problems tested that the proposed schemes derived for various values of k are of the same order, the block schemes gotten from the minimal value of k performed excellently and converges to the exact solution. Also the block schemes from $k = 3$ speed up the computational process than block schemes at $k = 2$

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