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A New Method for obtaining the n[th] Derivative of a Function of the Form $y = l_1(x)l_2(x)l_3(x)$

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Abstract

In this work, we use product rule and Leibnitz's theorem to generate a new method which can be used to obtain the higher order derivatives of any functions which can be written as $y = l_1(x)l_2(x)l_3(x)$. A Theorem that establishes the new method is presented proved by using mathematical induction.

The new method does not require the knowledge of the preceding derivative before obtaining the succeeding ones.

Keywords: product rule, Leibnitz's theorem, derivatives, numerical integrator, induction.

Introduction

The need for higher order derivatives of some functions are emerging in engineering, sciences and technology. The use of higher order derivatives of the interpolant involved is required in the implementation of some numerical integrators [1, 2, 3]. Thus, we present a new method for generating the higher order derivatives of functions that are dependent on three variables. Mathematical induction was used to prove the theorem that emerged from it.

1.0 Consider the function $y = l_1(x)l_2(x)l_3(x)$. (1)

Then the first derivative of $y(x)$ can be obtained as follows;

$$\frac{dy}{dx} = l_1(x)l_2(x)\frac{d}{dx}\{l_3(x)\} + l_3(x)\frac{d}{dx}\{l_1(x)l_2(x)\} \quad (2)$$

We proceed to substitute for

$$\frac{d}{dx}\{l_1(x)l_2(x)\}$$

its Leibnitz's theorem expression, such that;

$$\frac{dy}{dx} = l_1(x)l_2(x)\frac{d^1}{dx^1}l_3(x) + l_3(x)\left\{\sum_{\iota=0}^{n=1} C_\iota^1\left(\frac{d^{n-\iota}}{dx^{n-\iota}}l_1(x)\right)\left(\frac{d^\iota}{dx^\iota}l_2(x)\right)\right\} \quad (3)$$

Equation (3) can be written as;

$$\frac{dy}{dx} = \frac{d^0}{dx^0}(l_1(x)l_2(x))\frac{d^1}{dx^1}l_3(x) + \frac{d^0}{dx^0}(l_3(x))\left\{\sum_{\iota=0}^{n=1} C_\iota^1\left(\frac{d^{n-\iota}}{dx^{n-\iota}}l_1(x)\right)\left(\frac{d^\iota}{dx^\iota}l_2(x)\right)\right\} \quad (4)$$

If it is true for n=1, then it must be true for n=2:

2.0 From (2), one obtains:

$$\frac{d^2y}{dx^2} = l_1(x)l_2(x)\frac{d^2}{dx^2}l_3(x) + \left\{\left(\frac{d}{dx}l_1(x)l_2(x)\right)\left(\frac{d}{dx}l_3(x)\right)\right\} + l_3(x)\frac{d^2}{dx^2}l_1(x)l_2(x)$$
$$+\left\{\left(\frac{d}{dx}l_1(x)l_2(x)\right)\left(\frac{d}{dx}l_3(x)\right)\right\} \quad (5)$$

$$\frac{d^2y}{dx^2} = l_1(x)l_2(x)\frac{d^2}{dx^2}l_3(x) + 2\left\{\left(\frac{d}{dx}l_1(x)l_2(x)\right)\left(\frac{d}{dx}l_3(x)\right)\right\} + l_3(x)\frac{d^2}{dx^2}l_1(x)l_2(x) \quad (6)$$

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By using the Leibnitz's theorem expression for the derivatives in (6), one obtains:

$$= l_1(x)l_2(x)\frac{d^2}{dx^2}l_3(x) + 2\left\{\sum_{\iota=0}^{n=1} C_\iota^n\left(\frac{d^{n-\iota}}{dx^{n-\iota}}l_1(x)\right)\left(\frac{d^\iota}{dx^\iota}l_2(x)\right)\right\}\frac{d^1}{dx^1}l_3(x)$$

$$+l_3(x)\left\{\sum_{\iota=0}^{n=2} C_\iota^n\left(\frac{d^{n-\iota}}{dx^{n-\iota}}l_1(x)\right)\left(\frac{d^\iota}{dx^\iota}l_2(x)\right)\right\} \tag{7}$$

Thus it follows that;

$$\frac{d^2y}{dx^2} = \sum_{r=0}^{n=2} C_r^m\left[\left(\sum_{i=0}^{n-i} C_i^n\left[\frac{d^{n-i}l_1(x)}{dx^{n-i}}\right]\left[\frac{d^il_2(x)}{dx^i}\right]\right)\left[\frac{d^rl_3(x)}{dx^r}\right]\right] \tag{8}$$

3.0 If it is true for n=1, n=2, then it must be true for n=k:

$$\frac{d^ky}{dx^k} = l_1(x)l_2(x)\frac{d^k}{dx^k}l_3(x) + k\left(\frac{d}{dx}l_1(x)l_2(x)\right)\left(\frac{d^{k-1}}{dx^{k-1}}l_3(x)\right) +$$

$$\frac{k(k-1)}{2!}\left(\frac{d^2}{dx^2}l_1(x)l_2(x)\right)\left(\frac{d^{k-2}}{dx^{k-2}}l_3(x)\right) + \cdots$$

$$+\frac{k(k-1)(k-2)\ldots(k-(k-1))}{(k-1)!}\left(\frac{d^{k-1-i}}{dx^{k-1-i}}l_1(x)l_2(x)\right)\left(\frac{d^1}{dx^1}l_3(x)\right)$$

$$+\frac{k(k-1)(k-2)(k-3)\ldots(1)}{(k)!}\left(\frac{d^k}{dx^k}l_1(x)l_2(x)\right)\left(\frac{d^0}{dx^0}l_3(x)\right) \tag{9}$$

By Leibnitz's theorem

$$\frac{d^ky}{dx^k} = l_1(x)l_2(x)\frac{d^k}{dx^k}l_3(x) + k\left\{\sum_{\iota=0}^{n=1} C_\iota^n\left(\frac{d^{n-\iota}}{dx^{n-\iota}}l_1(x)\right)\left(\frac{d^\iota}{dx^\iota}l_2(x)\right)\right\}\frac{d^{k-1}}{dx^{k-1}}l_3(x)$$

$$\frac{k(k-1)}{2!}\left\{\sum_{\iota=0}^{n=2} C_\iota^{k-1}\left(\frac{d^{n-i}}{dx^{n-i}}l_1(x)\right)\left(\frac{d^\iota}{dx^\iota}l_2(x)\right)\right\}\frac{d^{k-2}}{dx^{k-2}}l_3(x) + \cdots$$

$$+\frac{k(k-1)(k-2)\ldots(k-(k-1))}{(k-1)!}\left\{\sum_{\iota=0}^{n=k-1} C_\iota^{k-1}\left(\frac{d^{n-i}}{dx^{n-i}}l_1(x)\right)\left(\frac{d^\iota}{dx^\iota}l_2(x)\right)\right\}\left(\frac{d}{dx}l_3(x)\right)$$

$$+\frac{k(k-1)(k-2)(k-3)\ldots(1)}{(k)!}\left\{\sum_{\iota=0}^{n=k} C_\iota^n\left(\frac{d^{n-\iota}}{dx^{n-\iota}}l_1(x)\right)\left(\frac{d^\iota}{dx^\iota}l_2(x)\right)\right\}\left(\frac{d^0}{dx^0}l_3(x)\right) \tag{10}$$

Thus, for n=k, the nth order derivative is;

$$\sum_{r=0}^{n=k} C_r^k\left[\left(\sum_{i=0}^{n=k-i} C_i^k\left[\frac{d^{k-i}l_1(x)}{dx^{k-i}}\right]\left[\frac{d^il_2(x)}{dx^i}\right]\right)\left[\frac{d^rl_3(x)}{dx^r}\right]\right] \tag{11}$$

4.0 If it is true for n=k, it must be true for n=k+1:

$$\frac{d^{k+1}y}{dx^{k+1}} = l_1(x)l_2(x)\frac{d^{k+1}}{dx^{k+1}}l_3(x) + (k+1)\left(\frac{d}{dx}l_1(x)l_2(x)\right)\left(\frac{d^k}{dx^k}l_3(x)\right) +$$

$$\frac{k(k+1)}{2!}\left(\frac{d^2}{dx^2}l_1(x)l_2(x)\right)\left(\frac{d^{k-1}}{dx^{k-1}}l_3(x)\right) + \cdots$$

$$+\frac{(k+1)k(k-1)\dots(3)}{(k-1)!}\left(\frac{d^{k-1}}{dx^{k-1}}l_1(x)l_2(x)\right)\left(\frac{d^2}{dx^2}l_3(x)\right)$$

$$+\frac{(k+1)k(k-1)(k-2)\dots(2)}{(k)!}\left(\frac{d^k}{dx^k}l_1(x)l_2(x)\right)\left(\frac{d^1}{dx^1}l_3(x)\right)$$

$$+\frac{(k+1)k(k-1)(k-2)\dots(1)}{(k+1)!}\left(\frac{d^{k+1}}{dx^{k+1}}l_1(x)l_2(x)\right)\left(\frac{d^0}{dx^0}l_3(x)\right) \qquad (12)$$

This leads to

$$\frac{d^{k+1}y}{dx^{k+1}} = l_1(x)l_2(x)\frac{d^{k+1}}{dx^{k+1}}l_3(x) + (k+1)\sum_{i=0}^{n=1}C_i^n\left[\frac{d^{n-i}l_1(x)}{dx^{n-i}}\right]\left[\frac{d^i l_2(x)}{dx^i}\right]\left(\frac{d^k}{dx^k}l_3(x)\right) +$$

$$\frac{k(k+1)}{2!}\sum_{i=0}^{n=2}C_i^n\left[\frac{d^{n-i}l_1(x)}{dx^{n-i}}\right]\left[\frac{d^i l_2(x)}{dx^i}\right]\left(\frac{d^{k-1}}{dx^{k-1}}l_3(x)\right) + \cdots$$

$$+\frac{(k+1)k(k-1)\dots(3)}{(k-1)!}\sum_{i=0}^{n=k-1}C_i^n\left[\frac{d^{n-i}l_1(x)}{dx^{n-i}}\right]\left[\frac{d^i l_2(x)}{dx^i}\right]\left(\frac{d^2}{dx^2}l_3(x)\right)$$

$$+\frac{(k+1)k(k-1)(k-2)\dots(2)}{(k)!}\sum_{i=0}^{n=k}C_i^n\left[\frac{d^{n-i}l_1(x)}{dx^{n-i}}\right]\left[\frac{d^i l_2(x)}{dx^i}\right]\left(\frac{d^1}{dx^1}l_3(x)\right)$$

$$+\frac{(k+1)k(k-1)(k-2)\dots(1)}{(k+1)!}\sum_{i=0}^{n=k+1}C_i^n\left[\frac{d^{n-i}l_1(x)}{dx^{n-i}}\right]\left[\frac{d^i l_2(x)}{dx^i}\right]\left(\frac{d^0}{dx^0}l_3(x)\right) \qquad (13)$$

Thus the $(k+1)^{th}$ *derivative is;*

$$\sum_{r=0}^{m=k+1}C_r^m\left[\left(\sum_{i=0}^{n=k}C_i^n\left[\frac{d^{n-i}l_1(x)}{dx^{n-i}}\right]\left[\frac{d^i l_2(x)}{dx^i}\right]\right)\left[\frac{d^r l_3(x)}{dx^r}\right]\right] \qquad (14)$$

In the above equations, $C_r^m = \dfrac{m!}{(m-r)!\,r!}$.

5.0 1^st^ Opeenoch's Theorem:

Let $y = l_1(x)l_2(x)l_3(x)$ or $y = l_1(x)l_2(x)/l_3(x)$

or $y = \frac{l_1(x)}{l_2(x)l_3(x)}$ *or* $y = \frac{1}{l_1(x)l_2(x)l_3(x)}$ or $y = l_1(x)l_2(x)/l_3(x)$.

Then the nth derivative of y is given as

$$\sum_{r=0}^{n=m} C_r^m \left[\left(\sum_{i=0}^{n=m-r} C_i^n \left[\frac{d^{n-i} l_1(x)}{dx^{n-i}}\right]\left[\frac{d^i l_2(x)}{dx^i}\right]\right)\left[\frac{d^r l_3(x)}{dx^r}\right]\right]$$

The coefficients of the above expression are obtained by the binomial theorem.

6.0 Conclusion:

The algorithm can easily be simulated by writing subroutines for the independent variables involved.

The following points are obvious concerning the new method:

(i) The superscript n decreases regularly by 1

(ii) The superscript i increases regularly by 1

(iii) The numerical coefficients are the normal binomial coefficients.

For increased accuracy in most numerical methods that involve the use of higher order derivatives, this new method can be used to obtain higher order derivatives of the functions involved [2, 3, 4]. The labour involved in calculating and evaluating higher derivatives through the use of this new method is very minimal, since you can jump the process of obtaining the preceding derivatives to the point of obtaining desired derivative (order).

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