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Cayley-Hamilton theorem in rhotrix

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Abstract

Cayley-Hamilton is one of the well-known theorems that is formulated and proved in linear algebra on matrices. In this paper we extend this theorem to the concept of rhotrix and also present some properties that are attached to it. Rhotrix is an object that lies in some way between $n\times n$ dimensional matrices and $(2n-1)\times(2n-1)$ dimensional matrices. Moreover, the representation of vectors in rhotrices is different from representation of vectors in matrices.

Keywords: Cayley-Hamilton, rhotrix, determinant, characteristic polynomial, minimum polynomial
AMS Subject Classifications [2010]:15A15, 15A18

1.0 Introduction

The concept of rhotrix was first introduced by Ajibade [1] as an extension of the initiative on matrix-tertions and matrix-noitrets suggested by Atanassov and Shannon [2]. The initial algebra and analysis of rhotrices was presented in [1]. The multiplication of rhotrices defined by Ajibade [1] is as follows: Let *R* and *Q* be two rhotrices such that

$$R=\left\langle \begin{matrix} & a & \\ b & h(R) & d \\ & e & \end{matrix} \right\rangle \text{ and } Q=\left\langle \begin{matrix} & f & \\ g & h(Q) & j \\ & k & \end{matrix} \right\rangle. \quad (1)$$

The addition and multiplication of rhotrices R and Q defined by Ajibade [1] are as follows:

$$R+Q=\left\langle \begin{matrix} & a+f & \\ b+g & h(R)+h(Q) & d+j \\ & e+k & \end{matrix} \right\rangle,$$

$$R\circ Q=\left\langle \begin{matrix} & ah(Q)+fh(R) & \\ bh(Q)+gh(R) & h(R)h(Q) & dh(Q)+jh(R) \\ & eh(Q)+kh(R) & \end{matrix} \right\rangle.$$

Another multiplication method for rhotrices called *row-column multiplication* was introduced by Sani [3] in an effort to answer some questions raised by Ajibade [1]. The row-column multiplication method is in a similar way as that of multiplication of matrices and is illustrated using the matrices *R* and *Q* defined in (1) as follows:

$$R\circ Q=\left\langle \begin{matrix} & af+dg & \\ bf+eg & h(R)h(Q) & aj+dk \\ & bj+ek & \end{matrix} \right\rangle.$$

A generalization of the row-column multiplication method for *n*-dimensional rhotrices was given by Sani [4]. That is: given *n*-dimensional rhotrices $R_n=\langle a_{ij},c_{lk}\rangle$ and $Q_n=\langle b_{ij},d_{lk}\rangle$ the multiplication of R_n and Q_n is as follows:

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$$R_n \circ Q_n = \left\langle a_{i_1 j_1}, c_{l_1 k_1} \right\rangle \circ \left\langle b_{i_2 j_2}, d_{l_2 k_2} \right\rangle = \left\langle \sum_{i_2 j_1 = 1}^{t} (a_{i_1 j_1} b_{i_2 j_2}), \sum_{l_2 k_1 = 1}^{t-1} (c_{l_1 k_1} d_{l_2 k_2}) \right\rangle,\ t = (n+1)/2. \qquad (2)$$

The method of converting a rhotrix to a special matrix called 'coupled matrix' was suggested by Sani [5]. This idea was used to solve systems of $n \times n$ and $(n-1)\times(n-1)$ matrix problems simultaneously. The concept of vectors, one-sided system of equations and eigenvector eigenvalue problem in rhotrices were introduced by Aminu [6]. A necessary and sufficient condition for the solvability of one sided system of rhotrix was also presented in [6]. If a system is solvable it was shown how a solution can be found. Rhotrix vector spaces and their properties were presented by Aminu [7]. Linear mappings and square root of a rhotrix were discussed by Aminu in [8] and [9] respectively.

To the author's knowledge Cayley-Hamilton theorem is not extended to rhotrices. It is the primary aim of this paper to extend this theorem to rhotrix and present some properties that are linked to it.

2. Rhotrix and its basic properties

Let $t = (n+1)/2$ for $n \in \mathbb{N}$. By 'rhotrix' we understand an object that lies in some way between $n \times n$ dimensional matrices and $(2n-1)\times(2n-1)$ dimensional matrices. That is an *n*-dimensional rhotrix is the following:

$$R_n = \left\langle a_{ij}, c_{lk} \right\rangle = \left\langle \begin{array}{ccccccc} & & & a_{11} & & & \\ & & a_{21} & c_{11} & a_{12} & & \\ & a_{31} & c_{21} & a_{22} & c_{12} & a_{13} & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{t1} & \cdots & \cdots & \cdots & \cdots & \cdots & a_{1t} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ & a_{t\,t-2} & c_{t-1\,t-2} & a_{t-1\,t-1} & c_{t-2\,t-1} & a_{t-2\,t} & \\ & & a_{t\,t-1} & c_{t-1\,t-1} & a_{t-1\,t} & & \\ & & & a_{tt} & & & \end{array} \right\rangle, \qquad (3)$$

Where $a_{ij}, c_{lk} \in \mathbb{R}$ for $i, j = 1,2,...,t$ and $k, l = 1,2...,t-1$. It is straightforward to verify that the addition of *n*-dimensional rhotrices $R_n = \left\langle a_{ij}, c_{lk} \right\rangle$ and $Q_n = \left\langle b_{ij}, d_{lk} \right\rangle$ is

$$R_n + Q_n = \left\langle a_{ij}, c_{lk} \right\rangle + \left\langle b_{ij}, d_{lk} \right\rangle = \left\langle \left(a_{ij} + b_{ij}\right), \left(c_{lk} + d_{lk}\right) \right\rangle, \qquad (4)$$

where $i, j = 1,2,...,t$ and $l, k = 1,2...t-1$ with $t = (n+1)/2$.
We will use throughout this paper the row-column multiplication method of rhotrices
defined in (2)

Rhotrix vectors (either row vectors or column vectors) can be represented in *t* different ways where $t = (n+1)/2$. This is different compared to vectors in matrices that can be represented in a unique way. For more information on rhotrix vectors the reader is referred to [6] and [7].

The *n*-dimensional identity rhotrix will be denoted by I_n and is given by

$$I_n = \left\langle \begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & 1 & 0 & & \\ & 0 & 0 & 1 & 0 & 0 & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ & 0 & 0 & 1 & 0 & 0 & \\ & & 0 & 1 & 0 & & \\ & & & 1 & & & \end{array} \right\rangle.$$

We will denote by 0 the usual zero, which is the neutral element under addition and for convenience we use the same symbol to denote any rhotrix or rhotrix vector whose every component is 0.

We will now summarize some basic properties of rhotrices that will be used later on. The following properties hold for *n*-dimensional rhotrices A, B and C over $\mathbb{R}$ and $\alpha \epsilon \mathbb{R}$:

$$A + 0 = 0 + A = A$$
$$A + B = B + A$$
$$(A + B) + C = A + (B + C)$$
$$\alpha(A + B) = \alpha A + \alpha B$$
$$A(B + C) = AB + AC$$
$$A(BC) = (AB)C$$
$$AI_n = A = I_n A$$

For an n-dimensional rhotrix $R_n = \langle a_{ij}, c_{lk} \rangle$ *the determinant* is defined as [4]

$$\det(R_n) = \det(A)\det(C)$$

Theorem 2.1. [4] *An n-dimensional rhotrix* $R_n = \langle a_{ij}, c_{lk} \rangle$ *is invertible if and only if the embedded matrices* $A = (a_{ij}) \epsilon \mathbb{R}^{t \times t}$ *and* $C = (c_{lk}) \epsilon \mathbb{R}^{t-1 \times t-1}$ *are invertible. Also if the inverse of A and C are denoted by* A^{-1} *and* C^{-1} *respectively, then the inverse of* R_n *is* $R_n^{-1} = \langle A^{-1}, C^{-1} \rangle$.

3. The rhotrix polynomial

Let $R_n = \langle a_{ij}, c_{lk} \rangle$ be an *n*-dimensional rhotrix defined in (2). The powers of R_n are defined as follows

$$R_n^2 = R_n R_n,\ R_n^3 = R_n^2 R_n, \ldots, R_n^{n+1} = R_n^n R_n \text{ and } R_n^0 = I_n.$$

Consider a polynomial $f(x) = a_0 + a_1 x = a_2 x^2 + \ldots + a_n x^n$, where a_i's are scalars in$\mathbb{R}$, we define polynomials in rhotrix as

$$f(R_n) = a_0 I_n + a_1 R_n = a_2 R_n^2 + \ldots + a_n R_n^n$$

Theorem 3.1. Let f and g be polynomials. Suppose $R_n = \langle a_{ij}, c_{lk} \rangle$ is an *n*-dimensional rhotrix and λ a scalar then

(i) $(f + g)R_n = f(R_n) + g(R_n)$ (iii) $(fg)R_n = f(R_n)g(R_n)$

(ii) $(\lambda f)(R_n) = \lambda f(R_n)$ (iv) $f(R_n)g(R_n) = g(R_n)f(R_n)$

Proof. The proof follows straightforwardly from definitions.

4. Adjoint of a rhotrix

Let $R_n = \langle a_{ij}, c_{lk} \rangle$ be an n-dimensional rhotrix, $A = (a_{ij}) \epsilon \mathbb{R}^{t\times t}$ and $C = (c_{lk}) \epsilon \mathbb{R}^{t-1\times t-1}$ the embedded matrices in R_n. Denote by A_{ij} the cofactor of a_{ij} and *the adjoint of* A denoted by $adjA$ is defined as

$$adjA = \left[A_{ij} \right]^T .$$

Similarly, let C_{lk} denote the cofactor of c_{lk} and $adjC$ *the adjoint of* C is define as

$$adjC = \left[C_{lk} \right]^T .$$

We define the adjoint of R_n as

$$adjR_n = \langle adjA, adjC \rangle.$$

Theorem 4.1. [10,11,12] Let A *be an* $n \times n$ *matrix then* $A(adj(A)) = (adj(A))A = \det(A) I_n$.

Theorem 4.2. Let $R_n = \langle a_{ij}, c_{lk} \rangle$ *be an n-dimensional rhotrix. Then*

$$R_n adj(R_n) = adj(R_n) R_n = \langle \det(A) I_t, \det(C) I_{t-1} \rangle$$

Proof. Let $R_n = \langle a_{ij}, c_{lk} \rangle$ be an n-dimensional rhotrix, from definitions we have $adjR_n = \langle adjA, adjC \rangle$. It follows from Theorem 4.1 that $adjA = adj(A) I_t$ and similarly, $adjC = adj(C) I_{t-1}$. Now we have from (2) and Theorem 4.1 that

$$\begin{aligned} R_n adj(R_n) = R_n(\langle adjA, adjC \rangle) &= \langle a_{ij}, c_{lk} \rangle (\langle adjA, adjC \rangle) \\ &= \langle A(adjA), C(adjC) \rangle \\ &= \langle (adjA)A, (adjC)C \rangle \\ &= \langle \det(A) I_t, \det(C) I_{t-1} \rangle, \end{aligned}$$

and the statement now follows.

Corollary 4.1. Let $R_n = \langle a_{ij}, c_{lk} \rangle$ *be an n-dimensional rhotrix. If* R_n *is invertible then*

$$R_n adj(R_n) = \det(R_n) \left\langle \tfrac{1}{\det(C)} I_t, \tfrac{1}{\det(A)} I_{t-1} \right\rangle$$

Proof. If $R_n = \langle a_{ij}, c_{lk} \rangle$ is invertible then it follows from Theorem 2.1 that the embedded matrices A and C in R_n are invertible. The statement now follows from Theorem 4.2.

5. The Cayley-Hamilton theorem

Suppose $R_n = \langle a_{ij}, c_{lk} \rangle$ is an n-dimensional rhotrix. The *indeterminate rhotrix* is defined as $N_n = \langle R_n - xI_n \rangle = \langle (a_{ij} - xI_t), (c_{lk} - xI_{t-1}) \rangle$, where I_n, I_t and I_{t-1} n-dimensional identity rhotrix, t-dimensional identity matrix and *(t-1)*-dimensional identity matrix respectively. The *characteristic polynomial* of R_n denoted by χ_{R_n} is defined as the determinant of the negative of indeterminate rhotrix, that is

$$\chi_{R_n}(x) = \det(xI_n - R_n).$$

Similarly, the characteristic polynomial of matrices *A* and *C* embedded in R_n denoted by χ_A and χ_C respectively are defined as:

$$\chi_A(x) = \det(xI_t - A) \text{ and } \chi_C(x) = \det(xI_{t-1} - C).$$

It is worthy to mention here that any square matrix is a root of its characteristic polynomial, in particular

$$\chi_A(A) = \chi_C(C) = 0. \tag{5}$$

Theorem 5.1. (Cayley-Hamilton) *Every rhotrix* $R_n = \langle a_{ij}, c_{lk} \rangle$ *is a root of its characteristic polynomial.*

Proof.

Let $R_n = \langle a_{ij}, c_{lk} \rangle$ be an arbitrary *n*-dimensional rhotrix and $\chi_{R_n}(x)$ be its characteristic polynomial

$$\chi_{R_n}(x) = \det(xI_n - R_n) = x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0.$$

From the definitions of determinant over rhotrix, we have

$$\begin{aligned}\chi_{R_n}(x) = \det(xI_n - R_n) &= \det(\langle xI_t - A, xI_{t-1}C \rangle) \\ &= \det(xI_t - A)\det(xI_{t-1} - C) \\ &= \chi_A(x)\chi_C(x)\end{aligned}$$

It follows straightforwardly from (2) and (5) that for $i, j = 1, 2, \ldots, t$ and $l, k = 1, 2, \ldots, t-1$

$$\begin{aligned}\chi_{R_n}(R_n) &= \chi_A(\langle a_{ij}, c_{lk} \rangle)\chi_C(\langle a_{ij}, c_{lk} \rangle) \\ &= \langle a_{ij}, 0 \rangle \langle 0, c_{lk} \rangle = \langle 0, c_{lk} \rangle \langle a_{ij}, 0 \rangle \\ &= \langle 0, 0 \rangle,\end{aligned}$$

where the first zero in the above rhotrices is a *t*-dimensional zero matrix while the second is a *t*-1-dimensional zero matrix with $t = (n+1)/2$.

6.0 An example

Consider the 3-dimensional rhotrix given in [6]:

$$R_3 = \left\langle \begin{matrix} & 4 & \\ 3 & 6 & 2 \\ & 3 & \end{matrix} \right\rangle.$$

The corresponding characteristics polynomial of R_3 is

$$\chi_{R_3}(x) = \det(xI_3 - R_3) = (x-6)^2(x-1) = x^3 - 13x^2 + 48x - 36.$$

Therefore,

$$\chi_{R_3}(R_3) = R_3^{\,3} - 13R_3^{\,2} + 48R_3 - 36I_3$$

$$\begin{aligned}\chi_{R_3}(R_3) &= \left\langle \begin{matrix} & 130 & \\ 129 & 216 & 86 \\ & 87 & \end{matrix} \right\rangle - \left\langle \begin{matrix} & 286 & \\ 273 & 468 & 182 \\ & 195 & \end{matrix} \right\rangle \\ &+ \left\langle \begin{matrix} & 192 & \\ 144 & 288 & 96 \\ & 144 & \end{matrix} \right\rangle - \left\langle \begin{matrix} & 36 & \\ 0 & 36 & 0 \\ & 36 & \end{matrix} \right\rangle = \left\langle \begin{matrix} & 0 & \\ 0 & 0 & 0 \\ & 0 & \end{matrix} \right\rangle\end{aligned}$$

Conclusion

In this paper we have successfully stated and proved one of the well-known theorems in linear algebra, which is the Cayley-Hamilton in rhotrix. Further research may concentrate on how this theorem can be applied.

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