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Rhotrices and Elementary Row Operations

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Abstract

In this note, elementary row operations on rhotrices are presented due to the vital roles they played in matrix theory. These operations can be used to determine rhotrix inverses and solve a system of n rhotrix equations.

Keywords: rhotrices; echelon rhotrices; elementary row operations

1.0 Introduction

The new paradigm of mathematical arrays of real numbers in rhomboid form called rhotrices was first introduced in [1] as an extension of matrix-tertions and matrix-noitrets suggested in [2] for mathematical enrichment.
Addition and scalar multiplication of rhotrices are like that of matrices as defined in [1]. Two rhotrices are multiplied together in the same way as two matrices by a method called row-column multiplication proposed in [3] as an alternative to heart-based multiplication method defined in [1].
Only the three-dimensional rhotrices which are referred to as base rhotrices were discussed in [1] and it was indicated that the dimension can be increased in size. As an extension to this, n-dimensional rhotrices were discussed in [4].The identity element, I_n and an inverse element Q_n of an n-dimensional rhotrix, $R_n = \langle a_{ij}, c_{lk} \rangle$ were given by

$$I_n = \left\langle \begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 0 & 1 & 0 & & & \\ & & 0 & 0 & 1 & 0 & 0 & & \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\ & & 0 & 0 & 1 & 0 & 0 & & \\ & & & 0 & 1 & 0 & & & \\ & & & & 1 & & & & \end{array} \right\rangle$$

and $Q_n = \langle q_{ij}, r_{lk} \rangle$ respectively, where (q_{ij}) is the inverse of (a_{ij}) and (r_{lk}) is the inverse of (c_{lk}).

Using the concept of rhotrix row and column vectors of base rhotrix defined in [3], the n-dimensional row and column vectors were defined in [5] as

$$\left\langle \begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & 0 & 0 & 0 & & & \\ & & 0 & 0 & 0 & 0 & 0 & & \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\ a_{t1} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\ & & a_{tt-2} & 0 & 0 & 0 & 0 & & \\ & & & a_{tt-1} & 0 & 0 & & & \\ & & & & a_{tt} & & & & \end{array} \right\rangle \quad \text{and} \quad \left\langle \begin{array}{ccccccccc} & & & & a_{11} & & & & \\ & & & a_{21} & 0 & 0 & & & \\ & & a_{31} & 0 & 0 & 0 & 0 & & \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\ a_{t1} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\ & & 0 & 0 & 0 & 0 & 0 & & \\ & & & 0 & 0 & 0 & & & \\ & & & & 0 & & & & \end{array} \right\rangle$$

respectively.

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It was indicated in [5] that there were *t* different ways of representing each of these vectors, where $t=(n+1)/2.$ It further indicated that the main rhotrix row and column vectors were represented as $\langle x^{in}\rangle$ and $\langle x^{nj}\rangle$ respectively, where $i, j = 1, 2,...,t$. The minor rows and columns were denoted by $\langle x^{\ln-1}\rangle$ and $\langle x^{n-1k}\rangle$ respectively, where $k, l = 1, 2,..., t-1$.

Therefore the necessary and sufficient condition for the solvability of the one-sided system of equations in rhotrices of the form

$$R_n\langle x^{nj}\rangle = b^{nj}$$

was discussed in [5], where R_n is an n-dimensional rhotrix, $\langle x^{nj}\rangle$ the unknown n-dimensional rhotrix vector and $\langle b^{nj}\rangle$ the right-hand-side rhotrix vector.

The aim of this article is to present the elementary row operations on rhotrices. Some of the vital applications of these operations such as the determination of rhotrix inverses and finding the solution to the system of equations in rhotrices are exposed.

2. Echelon Rhotrices

Definition 2.1: An n-dimensional rhotrix $R_n = \langle a_{ij}, c_{lk}\rangle$ is said to be echelon rhotrix or in echelon form, if the number of zeros preceding the first non-zero entry of a row increases row by row until only zero rows remain.

Remarks:

1. The first non-zero entry in each row is called distinguished element.
2. An echelon rhotrix is called rows reduced echelon rhotrix if the distinguished elements are:

(i) The only non-zero entries in their respective column

(ii) Each equal to 1

Example: $R_5 = \left\langle \begin{array}{ccccc} & & 2 & & \\ & 0 & -2 & 5 & \\ 0 & 0 & 3 & 0 & 1 \\ & 0 & 0 & -2 & \\ & & 0 & & \end{array} \right\rangle$ is an echelon rhotrix and $Q_5 = \left\langle \begin{array}{ccccc} & & 1 & & \\ & 0 & 1 & 0 & \\ 0 & 0 & 1 & 1 & 2 \\ & 0 & 0 & 3 & \\ & & 0 & & \end{array} \right\rangle$ is rows reduced echelon rhotrix.

Proposition 2.1

Let $R_n = \langle a_{ij}, c_{lk}\rangle$ be an n-dimensional rhotrix. Then R_n is an echelon rhotrix if and only if it's corresponding matrices (a_{ij}) and (c_{lk}) are echelon matrices.

Proof:

Suppose R_n is an echelon rhotrix. Then

$$R_n = \left\langle \begin{array}{ccccccccc} & & & & a_{11} & & & & \\ & & & 0 & c_{11} & a_{12} & & & \\ & & 0 & 0 & a_{22} & c_{12} & a_{13} & & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & a_{1t} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ & & 0 & 0 & 0 & c_{t-2t-1} & a_{t-2t} & & \\ & & & 0 & 0 & a_{t-1t} & & & \\ & & & & 0 & & & & \end{array} \right\rangle$$

$$\Leftrightarrow (a_{ij}) = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1t} \\ 0 & a_{22} & \cdots & a_{2t} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \ (c_{lk}) = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1t-1} \\ 0 & c_{22} & \cdots & c_{2t-1} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

$\Leftrightarrow (a_{ij})$ and (c_{lk}) are echelon matrices

Proposition 2.2

An n-dimensional rhotrix $R_n = \langle a_{ij}, c_{lk} \rangle$ is rows reduced echelon rhotrix if and only if it's corresponding matrices (a_{ij}) and (c_{lk}) are row reduced echelon matrices.

Proof:

Let R_n be a rows reduced echelon rhotrix. Then

$$R_n = \left\langle \begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 0 & 1 & 0 & & & \\ & & 0 & 0 & 1 & 0 & a_{13} & & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & a_{t1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ & & 0 & 0 & 0 & c_{t-tt-1} & a_{t-2t} & & \\ & & & 0 & 0 & a_{t-1t} & & & \\ & & & & 0 & & & & \end{array} \right\rangle$$

$$\Leftrightarrow (a_{ij}) = \begin{bmatrix} 1 & 0 & \cdots & a_{1t} \\ 0 & 1 & \cdots & a_{2t} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \ (c_{lk}) = \begin{bmatrix} 1 & 0 & \cdots & c_{1t-1} \\ 0 & 1 & \cdots & c_{2t-1} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

$\Leftrightarrow (a_{ij})$ and (c_{lk}) are row reduced echelon matrices

It worth noticing that every rhotrix can be transformed into row (column) echelon form, or into reduced rows (columns) echelon form by elementary row or column operations (ERO) on each of its corresponding matrices.
Recall that an elementary row (column) operation on a matrix M is any one of the following:

(i) Interchange the ith row and jth row. $R_i \leftrightarrow R_j$.

(ii) Multiply the ith row by a non-zero scalar K. $R_i \to KR_i$, $K \neq 0$.

(iii) Replace the ith row by K times the jth row plus the ith row. $R_i \to KR_j + R_i$.

Example: If

$$R_7 = \left\langle \begin{array}{ccccccc} & & & 0 & & & \\ & & 1 & 1 & 1 & & \\ & 2 & 2 & 3 & 0 & 3 & \\ 1 & -3 & 1 & 1 & -1 & -1 & 2 \\ & -7 & -1 & 4 & 1 & 0 & \\ & & 11 & 1 & 2 & & \\ & & & 4 & & & \end{array} \right\rangle$$

it is easy to verify that $\begin{bmatrix} 1 & 0 & -10 & -6 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 21 & 12 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & -3 \end{bmatrix}$

are the echelon forms of $\begin{bmatrix} 0 & 1 & 3 & 2 \\ 1 & 3 & -1 & 0 \\ 2 & 1 & 4 & 2 \\ 1 & -7 & 11 & 4 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 1 \\ -3 & -1 & 1 \end{bmatrix}$ respectively. Thus the echelon form of R_7 is

$$\left\langle \begin{array}{c} 1 \\ 0 \quad 1 \quad 0 \\ 0 \quad 0 \quad 1 \quad 0 \quad -10 \\ 0 \quad 0 \quad 0 \quad 1 \quad 3 \quad -1 \quad -6 \\ 0 \quad 0 \quad 21 \quad -1 \quad 2 \\ 0 \quad -3 \quad 12 \\ 0 \end{array} \right\rangle .$$

Similarly, $\begin{bmatrix} 1 & 0 & 0 & -\frac{2}{7} \\ 0 & 1 & 0 & \frac{2}{7} \\ 0 & 0 & 1 & \frac{4}{7} \\ 0 & 0 & 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & -3 \end{bmatrix}$ are row reduced echelon forms of $\begin{bmatrix} 0 & 1 & 3 & 2 \\ 1 & 3 & -1 & 0 \\ 2 & 1 & 4 & 2 \\ 1 & -7 & 11 & 4 \end{bmatrix}$ and

$\begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 1 \\ -3 & -1 & 1 \end{bmatrix}$ which imply that $\left\langle \begin{array}{c} 1 \\ 0 \quad 1 \quad 0 \\ 0 \quad 0 \quad 1 \quad 0 \quad 0 \\ 0 \quad 0 \quad 0 \quad 1 \quad 0 \quad -1 \quad -\frac{2}{7} \\ 0 \quad 0 \quad 1 \quad -1 \quad \frac{2}{7} \\ 0 \quad -3 \quad \frac{4}{7} \\ 0 \end{array} \right\rangle$ is the rows reduced echelon form of R_7 .

3. Applications of elementary row operations

3.1 Determination of rhotrix inverses

Let R_n be an n-dimensional rhotrix, and then its inverse R_n^{-1} if it exists can be determined by rhotrix row operations. This is to be achieved with following useful steps:

1. Form the augmented matrix by placing the identity matrix to the right of the corresponding matrix of R_n .
2. Use row operations to get 1 in row1, column1.
3. Get zeros in the rest of column1.
4. Continue step2 through step3 for the rest of rows and columns until the original matrix is changed in to the identity.
5. The matrix to the right of the bar is the desired inverse.

Example: If $R_5 = \left\langle \begin{array}{c} 1 \\ 1 \quad 1 \quad 3 \\ 1 \quad 3 \quad 4 \quad 2 \quad 3 \\ 3 \quad 4 \quad 3 \\ 4 \end{array} \right\rangle$ then $(a_{ij}) = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$, $(c_{lk}) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

One can verify that $\left[\begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 1 & 4 & 3 & 0 & 1 & 0 \\ 1 & 3 & 4 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 7 & -3 & -3 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$ so that

$(a_{ij})^{-1} = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$. Similarly $\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & 1 & \frac{3}{2} & -\frac{1}{2} \end{array}\right]$ implies that

$$(c_{lk})^{-1} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}. \text{ Thus } R_5^{-1} = \left\langle \begin{array}{ccccc} & & 7 & & \\ & -1 & -2 & -3 & \\ -1 & \frac{3}{2} & 1 & 1 & -3 \\ & 0 & -\frac{1}{2} & 0 & \\ & & 1 & & \end{array} \right\rangle.$$

3.2 Solution of one sided system of equations in rhotrices

In this subsection, we shall use rhotrix method to solve the one sided system of equations in rhotrices by elementary row operations.

Recall that in [5], a system of the form

$$R_n \left\langle x^{nj} \right\rangle = \left\langle b^{nj} \right\rangle, \qquad (1)$$

where R_n is an n-dimensional rhotrix, $\left\langle x^{nj} \right\rangle$ the unknown n-dimensional rhotrix vector and $\left\langle b^{nj} \right\rangle$ the right-hand-sided rhotrix vector is called a system of n rhotrix equations.

Moreover, the solvability of the system (1) depends on the solvability of the corresponding system of equations,

$$Ax^{tj} = b^{tj}, \qquad (2)$$

where $A = (a_{ij}) \in \Re^{t\,x\,t}$, $x^{tj}, b^{tj} \in \Re^{t\,x\,1}$ and $t = (n+1)/2$.

Also from [5], finding the solution of (1) is equivalent to find the solution x^{tj} of the system (2) and the corresponding $\left\langle x^{nj} \right\rangle$ is thus the solution of (1). Therefore solution to (2) can be determined by elementary row operations as follows:

1. Form the augmented matrix by placing b^{tj} to the right of the corresponding matrix *A*.
2. As in the case of the determination of rhotrix inverses, apply elementary row operations to the augmented matrix until the matrix *A* change to it's corresponding identity.

Example:

For the system $\left\langle \begin{array}{ccccc} & & 2 & & \\ & 1 & 2 & 5 & \\ 1 & 3 & 4 & 3 & 4 \\ & -3 & 6 & 3 & \\ & & -2 & & \end{array} \right\rangle \left\langle \begin{array}{ccccc} & & x_1 & & \\ & x_2 & 0 & 0 & \\ x_3 & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & \\ & & 0 & & \end{array} \right\rangle = \left\langle \begin{array}{ccccc} & & 4 & & \\ & 1 & 0 & 0 & \\ 5 & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & \\ & & 0 & & \end{array} \right\rangle$,

$A = \begin{bmatrix} 2 & 5 & 4 \\ 1 & 4 & 3 \\ 1 & -3 & -2 \end{bmatrix}$, $x^{31} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $b^{31} = \begin{bmatrix} 4 \\ 1 \\ 5 \end{bmatrix}$. Thus the augmented matrix is

$$\left[\begin{array}{ccc|c} 2 & 5 & 4 & 4 \\ 1 & 4 & 3 & 1 \\ 1 & -3 & -2 & 5 \end{array}\right].$$

Applying the required ERO to this augmented matrix we have $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{array}\right]$.

Therefore the solution to the system $Ax^{31} = b^{31}$ is $x^{31} = \begin{bmatrix} 3 \\ -2 \\ 2 \end{bmatrix}$ which implies that the solution to the given system of 5-rhotrix equation is the corresponding 5-dimensional rhotrix vector $\langle x^{51} \rangle = \left\langle \begin{array}{ccccc} & & 3 & & \\ & -2 & 0 & 0 & \\ 2 & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & \\ & & 0 & & \end{array} \right\rangle$.

3. Conclusion

In this note, we have discussed the elementary row operations on rhotrices and their applications. These operations have been applied in the determination of rhotrix inverses and finding the solution of the system of equations in rhotrices. In both cases the applications are buttressed by examples. The similarities exist between rhotrices and matrices in terms of additive and multiplicative operations made us to investigate in this new area of research (rhotrix theory) these results that were existed in matrix theory for mathematical enrichment.

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