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Rigorous Verification for the Solution of Nonlinear Interval System

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Abstract

We survey a general method for solving nonlinear interval systems of equations. In particular, we paid special attention to the computational aspects of linear interval systems since the bulk of computations are done during the stage of computing outer estimation of the including linear interval systems. The height of our findings is the synchronization of Hansen's theorem with that due to Rohn to accelerate basic convergence characteristics of our method. We compare computed results with those obtained by Sainz et al where Kaucher interval arithmetic was applied on interval Jacobi iterative type method and found out that our proposed method gave quite impressive results.

Keywords: nonlinear interval systems of equations, interval newton method, Hansen's method, Rohn's method, kaucher arithmetic.

1.0 Introduction

The paper considers the major steps to be taken when finding solution of nonlinear interval system of equation

$$F(x)=0 \qquad (1.1)$$

with $F : ID \subseteq IR^n \to IR^n$, F has at least $C^1 \in (ID)$, $[\underline{x}_i, \overline{x}_i]$ is an interval and

$x \in [x]$. $[x] = \prod_{i=1}^{n}([\underline{x}_i, \overline{x}_i])$ is an n dimensional Cartesian product often called a box in IR^n.

Computer arithmetic can be applied to rigorously verify the existence or absence of zeros in the equation 1.1 aided by the use of interval extensions and computational fixed point theorems. Such examples of fixed point theories include Contraction mapping theorem, Brouwer fixed point theorem and Miranda's theorem.

A contractor in interval arithmetic is a map that replaces a domain containing a solution set to equations of interest with a smaller one that also contains the solution set.

Miranda's theorem [1] asserts that : ''Supposing $x \in IR^n$, and assuming the faces of x be denoted by

$$x_i = \left(x_1, x_2, ..., x_{i-1}\ \underline{x}_i, x_{i+1}, ..., x_n \right)^T,$$

$$x_i = \left(x_1, x_2, ..., x_{i-1}, \overline{x}_i, ..., x_n \right)^T$$

Let $F = (f_1, f_2, ..., f_n)^T$ be a continuous function defined on **x. If**

$$\mathbf{f}_i^u(\underline{\mathbf{x}}_i)\mathbf{f}_i^u(\overline{x}_i) \le 0 \ ,$$

for each i between 1 and n, then there is an $x \in \mathbf{x}$ such that $F(\mathbf{x}) = 0$.''

Fundamentally, the nonlinear system (1.1) is transformed into equivalent linear interval system

$$F(x) = F(x_c) + J(x, x_c)(x - x_c), \qquad (1.2)$$

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$$x, x_c \in ID \quad \text{and} \quad J(x, x_c) = \left(\frac{\partial F_i}{\partial x_j}\left(x_c + \theta_i (x - x_c)\right) \right) \in IR^{n \times n}, i, j = 1,2,...,n.$$

Here $IR^{n\times n}$ denotes the real interval matrix, $\theta_i\,(i = 1,2,...,n)$ are some numbers lying between 0 and 1. Assuming that x^* approximates x very closely to the solution of problem (1.1).The interval Jacobian matrix is then split in the form

$J(x) = M(x) - N(x)$, M(x) and N(x) $\in IR^{n\times n}$. A convergent iterative sequence may be written as

$$x^* = x_c - M(x)^{-1}\left\{N(x)\left(x_c - x^*\right) + F(x_c)\right\} , \tag{1.3}$$

which may be compared with well known iteration of [2]. Hence forth, we signify A(x) to represent the Jacobian matrix J(x), the [b] represents the evaluation of F([x]). The splitting of J(x) will take the form

$$[A(x)]= [M(x)]-[N(x)]. \tag{1.4}$$

Basic interval methods for doing this is the well known IGA([A],[b]) called the interval Gaussian algorithm. Hence, we can develop an iteration of type

$$x^* = x_c - IGA\left([M(x)],[N(x)]\left(x_c - [x]^{(0)}\right) + b_c\right) , \; b_c \in [b] \; . \tag{1.5}$$

The Newtonian steps is written as

$$[x]^{(k+1)} = \left\{x^{(k)} - IGA\left([M(x)^{(k)}],[N(x)^{(k)}]\left(x_c^{(k)} - [x]^{(k)}\right) + b_c^{(k)}\right)\right\} \cap [x]^{(k)}, \tag{1.6}$$

K=0,1,2,.., and $x_c^{(k)} \in [x]^{(k)}$.

Various iterative processes below can be derived [3,4]:

$$[x]^{(k,0)} = [x]^{(k)} \quad , \tag{1.7}$$

$$[y]^{(k,m)} = x_c^{(k)} - IGA\left\{[M(x)]^{(k)},[N(x)]^{(k)}\left(x_c^{(k)} - [x]^{(k,m-1)}\right) + b_c^{(k)}\right\} , \tag{1.8}$$

$$[x]^{(k,m)} = [y]^{(k,m)} \cap [x]^{(k,r_k)} \quad , \tag{1.9}$$

$$[x]^{(k+1)} = [x]^{(k,r_k)} \quad , \tag{1.10}$$

Where $[x]^{(0)} \in IR^n ,$ and r_k is a sequence of integers.

Further splitting [5] of A(x) into

$$[A(x^{(k)})] = [D(x^{(k)})] - [B(x^{(k)})] - [C(x^{(k)})] \; , \tag{1.11}$$

could be obtained in a similar way and $[D(x^{(k)})]$ is the interval diagonal part of $[A(x^{(k)})]$. The terms $[B(x^{(k)})]$, respectively, $[C(x^{(k)})]$ are the strictly lower interval triangular and upper triangular matrices of $[A(x^{(k)})]$. The hierarchy to the generality of methods by combining equations (1.4) and (1.11) under focus can be derived in the form:
$[A(x^{(k)})] = F'([x^{(k)}]), [M(x^{(k)})] = [D(x^{(k)})], [N(x^{(k)})] = [B(x^{(k)})] + [C(x^{(k)})], r_k = 1,$ and r_k arbitrary, we have the case of Newton –Like single step method and its modification. Again setting
$[A(x^{(k)})] = [M(x^{(k)})] = F'([x^{(0)}]), [N(x^{(k)})] = 0, r_k = 1$, a simplified Newton's method is obtained.

Now using $[M(x^{(k)})] - [N(x^{(k)})]$ as triangular splitting of $[A(x^{(k)})]$ a modification of equations (1.9) and (1.10) is derived in the form:

$$[x]_i^{(k,m)} = \left\{ x_{c,i}^{(k)} - \left(\frac{1}{[m_{ii}(x^{(k)})]} \right) \left(\sum_{j=1}^{i-1} [m(x^{(k)})_{i,j} \left(x_{c,j}^{(k)} - [x]_j^{(k)}\right) + \left([N(x^{(k)})]\left(x_c^{(k)} - [x]^{(k,m-1)}\right)\right)_i + b_{c,i}^{(k)} \right) \right\} \cap [x]_i^{(k,m-1)}$$

$$, \quad (1 \le i \le n) \; . \tag{1.11}$$

The remaining sections in the paper have been arranged as follows.

Section 2 of the paper presents the methodology of approach to the problem wherein preconditioned interval system of equation (1.1) can be solved via generalized Hansen-Sengupta method . We took note of regularity of interval matrix A and necessary conditions for convergence are stated by synchronizing Hansen's theorem [6] with those of Rohn's theorems [7,8]. Section 3 in the paper gives numerical illustration with the methods under investigation. We concluded the paper based on the findings computed from these results with the methods.

2.0 Methodology

In the presentation of our methods we follow [8] as well as [9]. We also collaborate with the ideas presented in [10] which enable the construction of inverse interval matrix without using input interval data.

Definition 2.1

The mapping $G : ID \subseteq IR^n \to IR^n$ is called a contraction in $K \subseteq ID$ if G maps K into itself and if there exists $\beta \in R$ such that $0 \le \beta < 1$ for which

$$\|G(x) - G(y)\| \le \beta \|x - y\| \ \forall x, y \in K \qquad (2.1)$$

Here β is called a contraction factor of G (in K). Thus every contraction in K is Lipschitz continuous .

As a follow up to our discussion we have

Theorem 2.1, [11]. Let the mapping $G : ID \subseteq IR^n \to IR^n$ be a contraction in the convex set $K \subseteq ID$. Then G has exactly one fixed point $x^* \in K$ and the iteration $x^{(k+1)} = G\left(x^{(k)}\right)$ converges for $x^{(0)} \in K$ at least linearly to $x^{(*)}$. The relation

$$\left\|x^{(k+1)} - x^{(*)}\right\| \le \beta \left\|x^{(k)} - x^*\right\|$$

and

$$\frac{\left\|x^{(k+1)} - x^{(k)}\right\|}{1+\beta} \le \left\|x^{(k)} - x^{(*)}\right\| \le \frac{\left\|x^{(k+1)} - x^{(k)}\right\|}{1-\beta}$$

hold good ,in which case β denotes a contraction factor of G.

In what follows, writing $x_c, x^* \in x, x^* \in N(x, x_c), x^* \in H(x, x_c)$, the iteration defined by

$$x^{(0)} = x\,,$$
$$x^{(k+1)} = N\left(x^{(k)}, x_c^{(k)}\right) \cap x^{(k)} \quad \text{(k=0,1,2,..,)} \qquad (2.2)$$

and

$$x^{(0)} = x\,,$$
$$x^{(k+1)} = H\left(x, x_c\right), \text{ (k=0,1,...),} \qquad (2.3)$$

for suitable $x_c^{(k)} \in x^{(k)}$ defines convergent interval methods.

Equation (2.3) defines Hansen- Sengupta method. Thus for $x^\infty \ne \phi$, $z = x^\infty - x_c^\infty$ satisfies $z_c = 0$ and $z = H\left(x^\infty, x_c^\infty\right) - x_c^\infty = \Gamma\left(RA, -RF\left(x_c^\infty\right), z\right)$, $|z| \le (RA)^{-1} rad\left(-RF\left(x_c^\infty\right)\right) = 0$; R is a preconditioning matrix and is the expression defined in equation (2.3) while that equation (2.2) defines the mathematical formula written as equation (1.11). We assume that R is regular since RA is regular, and that, $\rho(R\Delta) < 1$ holds .We also take note that $\Delta = \frac{1}{2}\left(\bar{A} - \underline{A}\right)$ is the radius of interval matrix A. The Hull of interval linear system of equation (1.2) for which interval matrix A is regular is given by the equation

$$\sum(A, b) = \{d \in R^n : Ad = b, \ \ for\ some\ A \in \mathbf{A}, b \in \mathbf{b}\,. \qquad (2.4)$$

It is the smallest interval vector that contains $\sum(\mathbf{A}, \mathbf{b})$.

Theorem 2.2 , [6] .

Let $M = [\underline{M}, \bar{M}]$ where M=RA, r=Rb , assuming $\underline{M}^{-1} \ge 0,$ defining

$$s^{(i)} = \begin{cases} \bar{r} & for\ i = j \\ \max\left(-\underline{r}_j, \bar{r}_j\right) & for\ \ j \ne i,\ j = 1,2,\ldots,n \end{cases}$$

$$t^{(i)} = \begin{cases} \underline{r}_i & for \;\; j = i \\ \min\left(\underline{r}_j, \overline{r}_j \right) & for \;\; j \neq i, \;\; j = 1,2,\ldots,N \end{cases}$$

$$c_i = \frac{1}{2\left(M^{-1}\right)_{ii} - 1}$$

Then the hull of the interval linear system (1.2) is

$$M^H r = \left[\underline{d}, \overline{d} \right] \tag{2.5}$$

With

$$\underline{d}_i = \begin{cases} c_i e_i^T M^{-1} t^{(i)} & for \;\; \underline{d}_i \geq 0 \\ e_i^T \underline{M}^{-1} t^{(i)} & for \;\; \underline{d}_i < 0 \end{cases} \tag{2.6}$$

$$\overline{d}_i = \begin{cases} e_i^T M^{-1} s^{(i)} & for \;\; \overline{d}_i \geq 0 \\ c_i e_i^T M^{-1} s^{(i)} & for \;\; \overline{d}_i < 0 \end{cases} \tag{2.7}$$

and i=1,2,..,n , e_i is the unit vector whose i- th coordinate is 1 and all of whose coordinates are 0.
Introducing the following theorem we are able to solve the preconditioned system:
Theorem 2.3, [7]. Let M be inverse positive. The hull of equation (1.2) is

$$M^H r = [\underline{d}, \overline{d}] \;\; . \tag{2.8}$$

The following are defined in the form

$$\underline{d}_i = \min\left\{ \underline{d}_i, c_i \underline{d}_i \right\}$$

$$\overline{d}_i = \max\left\{ \overline{d}_i, c_i \overline{d}_i \right\}$$

and

$$\begin{aligned} \underline{d}_i &= -d_i^* + \left(M^{-1}\right)_{ii} \left(\overline{r} + \left|\overline{r}\right| \right)_i \\ \overline{d}_i &= d_i^* + \left(\underline{M}^{-1}\right)_{ii} \left(\overline{r} - \left|\overline{r}\right| \right)_i \end{aligned} \tag{2.9}$$

$$d_i^* = \left(\underline{M}^{-1} \left(\left|\overline{r}\right| + \frac{\left(\overline{r} - \underline{r} \right)}{2} \right) \right)$$

$$c_i = \frac{1}{2\left(M^{-1}\right)_{ii} - 1} \in (0,1) \tag{2.10}$$

r is the preconditioned interval vector of the system under consideration.
We synchronize the ideas given in [6] with that of [7,10] from which we are able to provide hull of solution set to the linear interval system of equation.

First using the fact that $I - G \approx I - R\Delta$, where $G = |I - RA_c| + |R|\Delta$, it is known [12] that $(I - |G|)^{-1} \approx (I - R\Delta)^{-1}$, and that $|R|\Delta \le (I - |I - RA_c|)^{-1}|R|\Delta$ remains valid. Next letting $M_0 = (I - |A_c^{-1}|\Delta)^{-1} \ge 0, R_0 = A_c^{-1}$, the $[A_c - \Delta, A_c + \Delta]^{-1}$ enclosure is given by the equation

$$[A_c - \Delta, A_c + \Delta]^{-1} = [R - (M - I)|R|, R + (M - I)|R|]. \qquad (2.11)$$

The convergence of the linear interval system is guided by the following considerations. R is the approximate inverse of A and I be the identity $n \times n$ matrix, $\|RA - I\|_\infty < 1$, implies A^{-1} exists and the computed approximate solution $d_c \in [d]$ satisfies error estimate

$$\|d_c - A^{-1}b\|_\infty \le \frac{\|R(Ad_c - b)\|_\infty}{1 - \|RA - I\|_\infty} \qquad (2.12)$$

3.0 Numerical Examples

Consider the following problem 1 taking from [13].

$$\begin{pmatrix} [4,6] & [-9,0] & [0,12] & [2,3] & [5,9] & [-23,-9] & [15,23] \\ [0,1] & [6,10] & [-1,1] & [-1,3] & [-5,1] & [1,15] & [-3,-1] \\ [0,3] & [-20,-9] & [12,77] & [-6,30] & [0,3] & [-18,1] & [0,1] \\ [-4,1] & [-1,1] & [-3,1] & [3,5] & [5,9] & [1,2] & [1,4] \\ [0,3] & [0,6] & [0,20] & [-1,5] & [8,14] & [-6,1] & [10,17] \\ [-7,-2] & [1,2] & [7,14] & [-3,1] & [0,2] & [3,5] & [-2,1] \\ [-1,5] & [-3,2] & [0,8] & [1,1] & [-5,10] & [2,7] & [6,82] \end{pmatrix} = \begin{pmatrix} [-10,95] \\ [35,14] \\ [-6,2] \\ [30,7] \\ [4,95] \\ [-6,46] \\ [-2,65] \end{pmatrix}$$

The following results in Table 1 were obtained [13] when dual interval arithmetic was applied on Jacobi iterative method . Let us note that a generalized interval $X \in KR$ (Kaucher arithmetic) is an interval whose bounds are not constrained to be ordered. For example $[-1,1] \in IR$ is a proper interval and $[1,-1] \in \overline{IR}$ is an improper one. Dual arithmetic is an example of this class, for example Dual[x,y]=[y,x].

Table 1. Results computed from problem 1 by [13] using Kaucher interval arithmetic on Jacobi iteration

Results X =	[-34.3722, 22.5419] [-11.6389, 26.3712] [-3.5476, 4.5974] [-18.0057, 27.0072] [-15.2489, 13.8566] [-31.9411, 4.2934] [-3.7336, 6.5295]

Table 2. Results of Problem 1 obtained when Hansen's theorem [6] is used in conjunction with Rohn's theorems, [7, 10].

Results X =	[-42.3264, 26.009] [-11.2591, 32.8422] [-3.8298, 5.0058] [-17.3622,32.9102] [-19.4266, 13.8837] [-27.1786, 15.2843] [-4.3883, 7.8786]

Table 3 Results of Problem 1 in floating point arithmetic

Results X =	-8.1627 10.7915 0.5880 7.7740 -2.7715 -5.9472 1.7451

4.0 Conclusion

We have presented the general over view in solving nonlinear interval systems. In particular, a greater attention was paid to the computational aspects of the resulting linear interval system since the bulk of work is done on the outer approximations in the including disks. As a result of this, a synchronization of [6] result was made with those obtained by [7, 10] which was used to accelerate some basic convergence characteristic behaviour in the computed results. We also compare notes with those of [13] where Kaucher interval arithmetic was applied on interval Jacobi iterative method. The big gain in our findings is that our proposed approach gave quite impressive results compare to those of [13] as revealed in Tables 1 and 2. Table 3 shows results computed where our computations were carried out in real floating point arithmetic. It follows that our proposed technique can be used to give worst case error bounds in computing problems where both linear and nonlinear issues are involved.

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