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Transformation of Groups

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Abstract

In this work, we have proved a number of purely geometric statements by algebraic methods. Also we have proved Sylvester's law of Nullity and Exercise: the nullity of the product BA never exceeds the sum of the nullities of the factor and is never less than the nullity of A.

Keywords: Transformation of Groups, Nullity, Kernel, Image, Non-Singular, Symmetry Group, Shear, Compression, Elongation Reflection
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1.0 Introduction

Molaei [5] studied the concept of generalized group where the group operation remained Crisp and the identity element of the group is not uniquely dependent on each element of the group and each element of the group has a unique inverse. In [3] however, Adeyemo took a departure from Molaei approach by considering the group structure and root system of SL_n over a field.

In the approaches of both Molaei and Adeyemo, the identity element of the group remained unique. Melhrabi et al [4] presented the concept of generalized subgroups and homomorphism. Interesting results about smooth generalized group, well Quasi-ordered sets and ideals in free semi groups and Algebras and lattice of congruences on a band of groups were obtained in [1], [6], and [7] respectively.

In this paper, we have proved a number of purely geometric statements by algebraic methods. Also we have proved Sylvester's law of Nullity and Exercise: *The nullity of the product BA never exceeds the sum of the nullities of the factor and is never less than the nullity of A.*

1.1 Theorem 1 (Sylvester's Law of Nullity)

The nullity of the product BA never exceeds the sum of the nullities of the factors and is never less than the nullity of A.

Proof

Let A: V→U, B: U→W. If $x\varepsilon$ker A, than BAx = 0, so that x is also in the kernel BA. Hence, Ker A< Ker BA. dim Ker A $\leq$ dim Ker BA. Hence the nullity of A.

Let $\{v_{1,\ldots,}v_t\}$ be a basis for the null space of A. Then by our earlier proof this can be extended to a basis $\{v_1,\ldots,v_t,x_{1,\ldots,}x_s\}$ of the null space of BA.

Let $x'_i = Ax_i,\ 1\leq i \leq s$

Suppose $0 = \sum a_i x_i$, then $A\left(\sum_1^s a_ix_i\right) = \sum_1^s a_iA(x_i) = \sum_1^s a_ix'_i = 0$. Hence since $\{x_1,\ldots,x_s\}$ is linearly independent, we see that $a_i = 0$, $1\leq i \leq s$. Hence $\{x'_1,\ldots,x'_s\}$ is linearly independent in U.

Moreover, $0 = BA(x_i) = Bx'i$, $1\leq i \leq s$. Thus $x'_1,\ldots,x'_s$ are linearly independent elements of Ker B, whence $s\leq n(B)$.

Thus $n(BA) = t + s = n(A) + s \leq n(A) + n(B)$.

Q.E.D.

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1.2 Theorem 2 (Fundamental Theorem of Linear Transformation)

If V has a finite dimension m and T: V→U is a linear transformation, then n(T) + r(T) = m.

Proposition 1

If A is a square matrix, then the nullity of BA is at least the nullity of B.

Proof

Let A be an mxm matrix, then B must be a pxm matrix for the product BA to be defined. Thus by theorem 2 above,

n(BA) = m – r(BA)

n(B) = m – r(B)

Let A: V→U, B: U→W. Suppose x' is an element of the image of V under BA, that is X' = (BA) (x) for some xεv. Then y = Ax lies in U and By = B(A(x)) = (BA) (x) = x'.

Hence imBA < imB; that is, r (BA) ≤ r (B). Whence m – r (B) ≤ m – r(BA). Thus n(B) ≤ n(BA).

1.3 Theorem 3

Any non-singular homogeneous linear transformation of the plane may be represented as a product of shears, one-dimensional compression (or elongation) and reflection.

1.4 Theorem 4

Any non-singular n x n matrix can be represented as a finite product of elementary matrices.

Remark

We shall make use of the following well-known theorem 4 in proving theorem 3.

Proof

We list all possible elementary 2×2 matrices and give the corresponding interpretation of each as a transformation of the plane.

$H_{12} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ (a reflection of the plane in the line y=x).

$M_1 = \begin{bmatrix} c & 1 \\ 0 & 0 \end{bmatrix}$ {a compression (or elongation) along the x-axis when c>0 and a compression (or elongation) followed by reflection in the y-axis if $c < 0$}

$M_2 = \begin{bmatrix} 1 & 0 \\ 0 & c \end{bmatrix}$ {a compression (or elongation) along the y-axis for c>0 or a compression (or elongation) along this y-axis followed by reflection in the x-axis if $c < 0$}

$F_{12} = \begin{bmatrix} 1 & 0 \\ d & 1 \end{bmatrix}$ {a shear parallel to the y-axis}

$F_{21} = \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix}$ {a shear parallel to the x-axis}

1.5 Theorem 5

The product of two linear transformations is linear

Proof:

By definition, a product TU maps any ξ into $\xi\{(TU)\} = (\xi T)U$. By the linearity of T and U respectively as in [2], $(c\xi + d\eta)TU = [c(\xi T) + d(\eta T)]U = c(\xi TU) + d(\eta TU)$, which is to any that TU also satisfies the defining condition for a linear transformation.

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1.6 Theorem 6

Any non-singular homogeneous linear transformation of the 3-dimensional real space may be represented as a product of two-dimensional shears, one-dimensional compression (or elongations), and reflection in planes.

Proof

We list all possible elementary 3 × 3 matrices and give the corresponding interpretation of each one as a homogeneous linear transformation of the 3-dimensional real space.

$H_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ {a reflection in the plane formed by the x-axis and the line y = z in the yz-plane}

$H_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ {a reflection in the plane formed by the z-axis and the line x = y in the xy-plane}

$H_{13} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ {a reflection in the plane formed by the y-axis and the line x = z in the xz-plane}

$M_1 = \begin{bmatrix} c & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ {a compression (or elongation) parallel to the x-axis when c>0 and a compression (or elongation) along the axis followed by reflection in the yz-plane if c>0}

$M_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & 1 \end{bmatrix}$ {a compression (or elongation) parallel to the y-axis when c>0 and a compression (or elongation) along the y-axis followed by a reflection in the xz-plane}

$M_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & c \end{bmatrix}$ {a compression (or elongation) parallel to the z-axis when c > 0 and compression (or elongation) along the z-axis followed by reflection in the xy-plane}

$F_{12} = \begin{bmatrix} 1 & 0 & 0 \\ d & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ {a shear in the xy-plane parallel to the y-axis}

$F_{21} = \begin{bmatrix} 1 & d & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ {a share in the xy-plane parallel to the x-axis}

$F_{13} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ d & 0 & 1 \end{bmatrix}$ {a shear in the xz-plane parallel to the z-axis}

$F_{31} = \begin{bmatrix} 1 & 0 & d \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ {a shear in the xz-plane parallel to the x-axis}

$F_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & d & 1 \end{bmatrix}$ {a shear in the yz-plane parallel to the z-axis}

$F_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & d \\ 0 & 0 & 1 \end{bmatrix}$ {a shear in the yz-plane parallel to the y-axis}

Conclusion

A set of transformation is said to form a group if it contains the inverse of each and the product of any two (including the product of one with itself or with its inverse). The number of distinct transformations is called the order of the group. (This may be either finite or infinite.) Clearly the symmetry operations of any figure form a group. This is called the symmetry group of the figure. In the extreme case where the figure is completely irregular (like the numeral six) its symmetry group is of order one, consisting of the identity alone.

Generally speaking, those one-one transformations of any set of elements which preserve any given property or properties of these elements form a group.

Felix Klein (Erlanger program 1872) has eloquently described how the different branches of geometry can be regarded as the study of those properties or suitable space which is preserved under appropriate group of transformation. The Euclidean geometry deals with those properties of space preserved under all isometries, and topology with those which are preserved under all homeomorphisms.

Similarly, projective and affine geometry deals with the properties which are preserved under the projective and affine group.

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