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Congruent Transformation

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Abstract

In this work, our main result will be a proof of ErdÖs-Mordell's theorem: If 0 is any point inside a triangle ABC and P, Q, R are feet of the perpendicular from 0 upon the respective sides BC, CA, AB, then OA + OB + OC > 2(OP+OQ+OP)

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1.0 Introduction

Chevalley [2] studied and obtained excellent results on invariant of finite groups generated by reflection. Stanley in [3] and [5] presented specialized cases of relative invariant of finite groups generated by pseudo-reflections and their application to combinatorics.

In [4] however, Cohen took a departure from Chevalley's approach by considering the finite complex reflection groups. In the approaches of both Linder and steedly [7] and Watanable [6] the symmetry operations of any figure form a group with those properties of space preserved under all isometries.

In this work, we shall study the congruent transformation from both the geometric and algebraic point of view. In particular, we shall give alternative proof to some simple theorems – the first being a geometrical proof and the second being an algebraic proof. In almost all the cases, the algebraic proof is simpler but often less instructive.

We have also given a pure geometric proof of ErdÖs Mordell's theorem following hints given by Coxeter [1]: If O is any point inside a triangle ABC and P ,Q ,R are the feet of the perpendicular from O upon the respective sides BC, CA, AB then OA + OB + OC $\geq$ 2(OP + OQ + OR).

0.2 NOTES ON TERMINOLOGY

1. Definition: A transformation T : X→ X where (X,d_1) and (Y, d_2) are metric spaces is called isometry if $d_1(x, y) = d_2(T(x), T(y))$.
2. Definition : A point x in a space is said to be invariant under a transformation T: X → X if T(x) = x.
3. Definition: If L,V are vector spaces we call a transformation T:L→ V homogenous if T (0) = 0, where 0 is the zero vector.

But first let us give two alternative proofs of:

1 .1 Prepositions (i) the reflection in the y-axis reverses the sign of x.

(ii) the reflection in the line x = y interchanges x and y

Proof: A Geometric proof

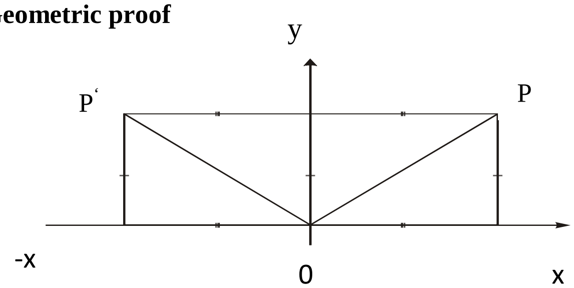

Fig. 1: Reflection in y-axis

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Let p be taken without loss generality, to lie in the first quadrant. Then the mirror image of P is point P' in the second quadrant obtained by producing the perpendicular from P to the y-axis, a distance equal to the length of this perpendicular.

Hence it follows that y(P') = y(P) and x(P') = - (P).

(i)

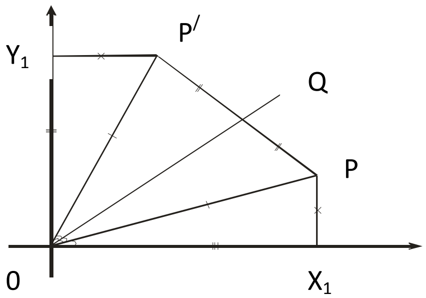

Fig. 2: Reflection in the line x = y

With the notations in the figure, OQ is perpendicular to the line segment PP' and if perpendicular to PP', it bisects it. Hence PQ = QP' and it also bisects P'ÔP. Whence P'ÔQ = QÔP. Op = op' opposite sides of isosceles triangle and OQ is common. Hence triangles OP'Q and OQP are congruent. Also since OQ bisects $y_1\hat{O}x_1$, then angles $y_1\hat{O}Q = Q\hat{O}x_1 = 45^0$. Hence angles $y_1\hat{O}Q$ – P'ÔQ = $Q\hat{O}x_1$ – QÔP. Whence angle $P\hat{O}x_1 = P'\hat{O}y_1$ and OP' = OP.

Triangles OPx_1 and $OP'y_1$ are congruent.

An algebraic proof

(i) This transformation can, relative to the standard basis $\{e_1, e_2\}$ of R^2, be represented by the matrix $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

Thus, for any vector v = (x, y) in R^2, we have

$$T_V = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ y \end{bmatrix}$$

This proves that reflection in the y-axis changes the sign of x

(ii) The transformation can be represented by the matrix

$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ relative to the standard basis $\{e_1, e_2\}$ of R_2. Thus for any vector v(x, y) in R_2, we have

$$T_v = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix}$$

Hence the reflection in the line x = y interchanges x and y.

1.2 **Lemma**: let f: $R^+ \rightarrow R$ be defined by $f(t) = t + \frac{1}{t}$, then f has a minimum at t = 1.

Proof : $f(t) = t + \frac{1}{t}$

$f'(t) = 1 - \frac{1}{t^2}$

$f'(t) = 0 \Rightarrow t^2 = 1 \Rightarrow t = \pm 1$

Substituting value of t, when t = -1, f''(t) = -2 and when t = 1, f''(t) = 2.

Hence t =1 corresponds to a minimum of f(t) as claimed,

Q.E.D.

1.3 Theorem 1 (Erd*ö*s-Mordell)

If O is any point inside a triangle ABC and P, Q, R are the feet of the perpendicular from O upon the respective sides BC, CA, AB, then OA +OB+OC ≥ 2(OP+OQ+OR)

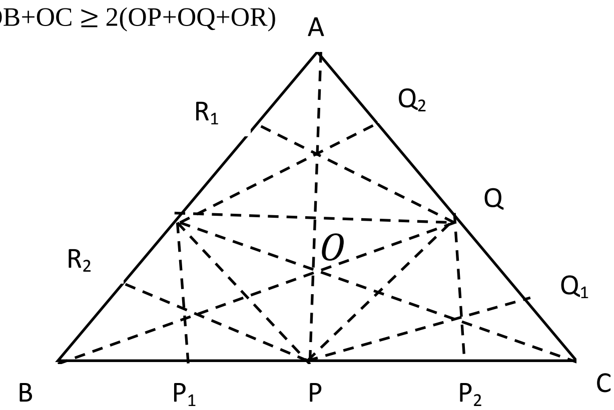

Fig. 3: Isosceles Triangle ABC with centre 0

Proof:

We first establish that: ΔPRP_1 is similar to ΔOBR sine points P, O, R, B lie on the circle with BO as the diameter, and angle at P_1 = angle R (right angles) and angle at P = angle at O (angles on the same side on an arc of a circle). Hence angle R = angle B.

Whence, $\Delta BOR = \Delta P_1PR$. Since $B\hat{R}O = R\hat{P}_1P$ right angles,

then $\frac{P_1P}{OR} = \frac{P_1P}{RB} = \frac{PR}{OB}$, whence ΔPRP_1 is similar to ΔOBR.

From $\frac{P_1P}{OR} = \frac{PR}{OB}$, we obtain $P_1P = \frac{OR \cdot RP}{OB}$.

Other analogous expressions are obtained from

$\Delta PQP_2 \sim \Delta OCP \cdot \frac{PP_2}{OQ} = \frac{QP}{OC} \Longrightarrow PP_2 = \frac{OQ \cdot QP}{OC}$. ΔQPQ_1 is similar to $\Delta OCP \cdot \frac{QQ_1}{OP} = \frac{QP}{OC} \Longrightarrow QQ_1 = \frac{OR \cdot QR}{OC}$, $\Delta QRQ_2 \sim \Delta OAR$.

$$\frac{QQ_2}{OR} = \frac{QR}{OA} \Longrightarrow QQ_2 = \frac{OR \cdot QR}{OA}, \Delta RQR_1 \sim \Delta OAQ.$$

$$\frac{RR_1}{OQ} = \frac{RQ}{OA} \Longrightarrow RR_1 = \frac{OQ \cdot QR}{OA}, \Delta RPR_2 \sim \Delta OBP.$$

$$\frac{RR_2}{OP} = \frac{RR_2}{OB} \Longrightarrow RR_2 = \frac{OP \cdot PR}{OB}$$

Noting that $P_1P_2 = P_1P + PP_2 \leq RQ \Longrightarrow \frac{P_1P + PP_2}{RQ} \leq 1$

$Q_1Q_2 = Q_1Q + QQ_2 \leq PR \Longrightarrow \frac{Q_1Q + QQ_2}{PR} \leq 1$

$R_1R_2 = R_1R + RR_2 \leq QP \Longrightarrow \frac{R_1R + RR_2}{QP} \leq 1$

Thus $OA = OA \cdot 1 \geq \frac{OA(P_1P + PP_2)}{RQ} = \frac{OA}{RQ}\left(\frac{OR \cdot RP}{OB} + \frac{OQ \cdot QP}{OC}\right)$

$OB = OB \cdot 1 \geq \frac{OB\,(Q_1Q + QQ_2)}{PR} = \frac{OB}{PR}\left(\frac{OP \cdot PQ}{OC} + \frac{OR \cdot RQ}{OA}\right)$

$OC = OC \cdot 1 \geq \frac{OC(R_1R + RR_2)}{QP} = \frac{OC}{QP}\left(\frac{OQ \cdot QR}{OA} + \frac{OP \cdot PR}{OB}\right)$

Whence $OA + OB + OC \geq OP\left\{\frac{OC}{QP} \cdot \frac{PR}{OB} + \frac{OB}{PR} \cdot \frac{PQ}{OC}\right\} +$

$OQ\left\{\frac{OC}{QP} \cdot \frac{QR}{OA} + \frac{OA}{RQ} \cdot \frac{QP}{OC}\right\} + OR\left\{\frac{OB}{PR} \cdot \frac{RQ}{OA} + \frac{OA}{RQ} \cdot \frac{RP}{OB}\right\}$

If we put $a = \frac{OB \cdot PQ}{OC \cdot PR}$, $b = \frac{OC}{OA} \cdot \frac{QR}{QP}$, $c = \frac{OA}{OB} \cdot \frac{RP}{RQ}$,

then $a > 0, b > 0, c > 0$. Moreover we have

$OA + OB + OC \geq OP\left(a + \frac{1}{a}\right) + OQ\left(b + \frac{1}{b}\right) + OR\left(c + \frac{1}{c}\right)$.

Furthermore, if $t > 0$, then $t + \frac{1}{t} \geq 2$ the minimum occurring when $t = 1$ by the above lemma.

Thus $OA + OB + OC \geq 2(OP + OQ + OR)$.

1.4 Theorem 2

If $T: R^2 \rightarrow R^2$ is an isometry which has more than one invariant point, then it must be either an identity or reflection.

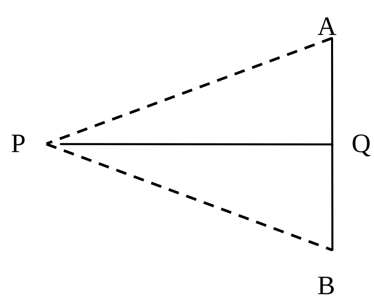

Fig. 4: Two invariant points in the plane

Proof

Let A and B be two distinct invariant points and let P be any point of the plane. If P' = T (P), then by hypothesis AP=AP', BP=BP'

Hence P' lies on the intersection of the circle with centre A and radius AP and that with centre B and radius BP.

From the triangle APB, we see that BP + AP ≥ AB (triangle inequality), and equality holds if and only if P lies on the straight line AB. Hence the circles touch each other if and only if P is on the line AB and they touch at P in that case.

Thus under T if P is on the line AB, P' = T(P) = P. If P is not on AB, then we see that the circles meet at two distinct points P and Q, If P' = Q, then T is the reflection along the line AB since the radius from A bisects the chord PQ orthogonally.

Q.E.D.

Conclusion

It is now convenient to use the word transformation in the special sense of one-one correspondence P→P' among all the points in the plane (or in space), that is a rule for associating a pair of points, with the understanding that each pair has a first member of just one pair and also has the second of just one pair. It may happen that the members of a pair coincide, that is, the P' coincide with P, in this case P is called an invariant point (or "double point") of the transformation.

In particular, an isometry (or "congruent transformation" or "congruence") is a transformation which preserves length, so that if (P,P') and (Q,Q') are two pairs of corresponding points, we have PQ = P'O' : PQ and P'Q' are congruent segment. For instance, a rotation of the plane about P (or about a line through P perpendicular to the plane) is an isometry having P as an invariant point, but a transformation (or "parallel displacement") has no invariant point: every point is moved.

A reflection is the special kind of isometry in which the invariant point consist of all points on a line (or plane) called mirror.

A still simpler kind of transformation is the identity, which leaves every point unchanged. The result of applying several transformations is the identity; each is called the inverse of the other and their product in the reverse order is again the identity.

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