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Harmonic Function of Poincare Cone Condition In Solving Dirichlet Problem

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Abstract

This paper describes the set of harmonic functions on a given open set U which can be seen as the kernel of the Laplace operator and is therefore a vector space over R .It also reviews the harmonic theorem, the dirichlet problem and maximum principle where we conclude that the application of sums , differences and scalar multiples of harmonic functions are again harmonic.

Keywords: Harmonic, maximum principles, Vector space, Poincare cone, Holder condition.

1.0 Introduction

In mathematics, mathematical physics and the theory of stochastic processes, a Harmonic function is a twice continuously differentiable function $f : U \to R$ (where U is an open subset of R^n) which satisfies Laplace's equation, i.e $\frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} + ... + \frac{\partial^2 f}{\partial x_n^2} = 0$ everywhere on U. This is usually written as $\nabla^2 f = 0$, an harmonic function defined on an annulus [1].

Harmonic functions are determined by their singularities. The singular points of the harmonic functions are expressed as "charges" and "charge densities" using the terminology of electrostatics, and so the corresponding harmonic function will be proportional to the electrostatic potential due to these charge distributions. Each function will yield another harmonic function when multiplied by a constant, rotated, and/or has a constant added. The inversion of each function will yield another harmonic function which has singularities which are the images of the original singularities in a spherical "mirror". Also, the sum of any two harmonic functions will yield another harmonic function [1, 2].

If f is a harmonic function on U, then all partial derivatives of f are also harmonic functions on U. The Laplace operator and the partial derivative operator will commute on this class of functions. The harmonic functions are real analogues to holomorphic functions. All harmonic functions are analytic, i. e, they can be locally expressed as power series [3].

The uniform limit of a convergent sequence of harmonic functions is still harmonic. This is true because any continuous function satisfying the mean value property is harmonic. Consider the sequence on $(-\infty, 0) \times R$ defined by $f_n(x, y) = \frac{1}{n}\exp(nx)\cos(ny)$. This sequence is harmonic and converges uniformly to the zero function; however note that the partial derivatives are not uniformly convergent to the zero function.

Subharmonic Functions

A C^2 function that satisfies $\nabla f \geq 0$ is called subharmonic. This condition guarantees that the maximum principle will hold, although other properties of harmonic functions may fail. More generally, a function is subharmonic if and only if, in the interior of any ball in its domain, its graph lies below that of the harmonic function interpolating its boundary values on the ball [1].

Let U be a **domain**, i.e a connected open set $U \subset \mathrm{R}^d$, and ∂U be its boundary. Suppose that its closure $\overline{U}$ is a homogeneous body and its boundary is electrically charged, the charge given by some continuous function $\phi : \partial U \to \mathrm{R}$. The Dirichlet problem asks for the voltage $u(x)$ at some point $x \in U$. Kirchhoff's laws state that u must be a harmonic function in U .

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Basic features of Harmonic Function

Let $U \subset \mathrm{R}^d$ be a domain. A function $u :\to R$ is harmonic (on U) if it is twice continuously differentiable and, for any $x \in U$,

$$\nabla u(x) := \sum_{j=1}^{d} \frac{\partial^2 u}{\partial_j^2}(x) = 0. \qquad (1)$$

if instead of the last condition only $\nabla u(x) \geq 0$, then the function u is called **subharmonic**.

We have two useful reformulations of the harmonicity condition, called the **mean value properties**, which do not make explicit reference to differentiability.

THEOREM 1: Let $U \subset \mathrm{R}^d$ be a domain and let $u : U \to R$ be measurable and locally bounded. The following conditions are equivalent:

(i) u is harmonic;

(ii) For any ball $B(x,r) \subset U,\ u(x) = \frac{1}{L(B(x,r))} \int_{\mathbf{B}(x,r)} u(y)dy;$

(iii) For any ball $B(x,r) \subset U,\ u(x) = \frac{1}{\sigma_{x,r}(\partial B(x,r))} \int_{\partial \mathbf{B}(x,r)} u(y) d\sigma_{x,r}(y),$

where $\sigma_{x,r}$ is the surface measure on $\partial B(x,r)$.

we use following version of Green's identity,

$$\int_{\partial \mathbf{B}(x,r)} \frac{\partial u}{\partial n}(y) d\sigma_{x,r}(y) = \int_{\mathbf{B}(x,r)} \Delta u(y) dy, \qquad (2)$$

where n(y) is the outward normal of the ball at y. The result can also be proved by purely probabilistic means.

Proof. (ii) $\Rightarrow$ (iii)

Assume u has the mean value property (ii). Define $\psi : (0, \infty) \to R$ by

$$\psi(r) = r^{1-d} \int_{\partial \mathbf{B}(x,r)} u(y) d\sigma_{x,r}(y)$$

we show that ψ is constant. Indeed, for any $r > 0,$

$$r^d L(\beta(x,1))u(x) = L(\beta(x,r))u(x) = \int_{\beta(x,r)} u(y)dy = \int_0^r \psi(s) s^{d-1} ds. \qquad (3)$$

Differentiating with respect to *r* gives $dL(\beta(x,1))u(x) = \psi(r),$ and therefore $\psi(r)$ is constant.

Now (iii) follows from the well-known identity $dL(\beta(x,r))/r = \sigma_{x,r}(\partial\beta(x,r)).$

(iii) $\Rightarrow$ (ii)

Fix $s > 0,$ multiply (ii) by $\sigma_{x,r}(\partial\beta(x,r))$ and integrate over all radii $0 < r < s.$

(iii) $\Rightarrow$ (i)

Suppose $g : [0,\infty) \to [0,\infty)$ is a smooth function with compact support in $[0,\varepsilon)$ and $\int g(|x|)dx = 1.$ Carying out the integration in the condition (iii) of Theorem 1, one obtains

$$u(x) = \int u(y) g(|x-y|) dy \qquad (4)$$

For all $x \in U$ and sufficiently small $\in > 0$. As convolution of a smooth function with a bounded function produces a smooth function, we observe that u is infinitely differentiable in U.

Now suppose that $\Delta u \neq 0,$ so that there exists a small ball $\beta(x,\varepsilon) \subset U$ such that either $\Delta u(x) > 0,$ or $\beta(x,\varepsilon),$ on $\Delta u(x) < 0,$ or $\beta(x,\varepsilon).$ Using the notation from above, we obtain that

$$0 = \psi^1(r) = r^{1-d} \int_{\partial\beta(x,r)} \frac{\partial u}{\partial n}(y) d\sigma_{x,r}(y) = r^{1-d} \int_{\beta(x,r)} \Delta u(y) dy.$$

Using eq.(2), this is a contradiction.

(iii) $\Rightarrow$ (i)

Suppose that u is harmonic and $\beta(x,r) \subset U$. With the notation from above and (2), we obtain that

$$\psi^1(r) = r^{1-d}\int_{\partial\beta(x,r)} \frac{\partial u}{\partial n}(y)d\sigma_{x,r}(y) = r^{1-d}\int_{\beta(x,r)}\Delta u(y)dy = 0 \tag{5}$$

Hence ψ is constant, and as $\lim_{r\downarrow 0}\psi(r) = \sigma_{0,1}(\beta(0,1))u(u)$, we obtain (iii)

Conclusively, twice differentiable function $u : U \to R$ is subharmonic if and only if

$$u(x) \le \frac{1}{L(\beta(x,r))}\int_{\beta(x,r)} u(y)\,dy \text{ for any } \beta(x,r) \subset U \tag{6}$$

This can be obtained in a way very similar to Theorem 1.
Hence, a harmonic function satisfies this important property, and in fact subharmonic functions satisfy the maximum principle.

Dirichlet Problem

Given a function *f* that has values everywhere on the boundary of a region in R^n, is there a unique continuous function *u* twice continuously differentiable in the interior and continuous on the boundary such that *u* is harmonic in the interior and *u = f* on the boundary.
For a domain D having a sufficiently smooth boundary ∂D, the general solution to the Dirichlet problem is given by

$$u(x) = \int_{\partial D} v(s)\frac{\partial G(x,s)}{\partial n}ds \tag{7}$$

where *G(x,y)* is the Green's function for the partial differential equation, and

$$\frac{\partial G(x,s)}{\partial n} = \hat{n}.\nabla_s G(x,s) = \sum_i n_i \frac{\partial G(x,s)}{\partial s_i}$$

is the derivative of the Green's function along the inward-pointing unit normal vector $\hat{n}$. The integration is performed on the boundary, with measure *ds*. The functions *v(s)* is given by the unique solution to the Fredholm integral equation of the second kind,

$$f(x) = -\frac{v(x)}{2} + \int_{\partial D} v(s)\frac{\partial G(x,s)}{\partial n}ds. \tag{8}$$

The Green's function to be used in eq (8) above integral is one which vanishes on the boundary *G(x,s) = 0*
For $s \in \partial D$ and $x \in D$. Such a Green's function is usually a sum of the free-field Green's function and a harmonic solution to the differential equation.
The Dirichlet problem was posed by Gauss [4]. In fact, Gauss [4] thought he showed that there is always a solution, but his reasoning was wrong and Lebesque and Zaremba[5, 6] gave counter examples. However, if the domain is sufficiently nice there is a solution, as we will see below.

Definition: Let U be a domain in R^d and let ∂U be its boundary. Suppose $\phi : \partial U \to R$ is a continuous function on its boundary. A continuous function $u : \overline{U} \to R$ is a **solution to the Dirichlet problem** with boundary value ϕ, if it is harmonic on U and $u(x) = \phi(x)$ for $x \in \partial U$ [7].

Definition: Let $U \subset R^d$ be a domain. We say that U satisfies the **Poincare cone condition** at $x \in \partial U$ [8].if there exists a cone V based at x with opening angle $\alpha > 0$, and $h > 0$ such that $V \cap \beta(x,h) \subset U^c$.
The following lemma will prepare us to solve the Dirichlet problem for 'nice' domains. Recall that we denote, for any open or closed set $A \subset R^d$, by $\tau(A)$ the first hitting time of the set A by Brownian motion, $\tau(A) = \inf\{t \ge 0 : B(t) \in A\}$.

Lemma 1: Let $0 < \alpha < 2\pi$ and $C_0(\alpha) \subset R^d$ is a cone base at the origin with opening angle α, and

$$\alpha = \sup_{x\in\beta\left(0,\frac{1}{2}\right)} P_x\left\{\left(\tau\partial\beta(0,1)\right) < \left(C_0(\alpha)\right)\right\}. \tag{9}$$

Then $\alpha < 1$ and, for any positive integer k and $h^1 > 0$, we have

$$P_x\left\{\left(\tau\partial\beta(z,h^1)\right) < \tau\left(C_z(\alpha)\right)\right\} \le a^k,$$

For all $x, z \in R^d$ with $|x - z| < 2^{-k}h^1$, where $C_z(\alpha)$ is a cone based at z with opening angle α.

Proof. Obviously $\alpha < 1$. If $x \in \beta(0, 2^{-k})$ then by the strong Markov property[8]

$$P_x\left\{\left(\varpi\partial\beta(0,1)\right)<\tau\left(C_z(\alpha)\right)\right\}$$
$$\le\prod_{i=0}^{k-1}\sup_{x\in\beta\left(0,2^{-k+i}\right)}P_x\left\{\tau\left(\partial\beta\left(0,2^{-k+i+1}\right)\right)<\tau\left(C_0(\alpha)\right)\right\}=a^k. \quad (10)$$

Therefore, for any positive integer k and $h^1>0$, we have by scaling $P_x\left\{\tau\left(\partial\beta\left(z,h^1\right)\right)<\tau\left(C_z(\alpha)\right)\right\}\le a^k$, for all x with $|x-z|<2^{-k}h^1$.

Theorem 2: *(Dirichlet Problem)*. Suppose $U\subset R^d$ is a boundary domain such that every boundary point satisfies the Poincare cone condition, and suppose ϕ is a continuous function on ∂U. Let $\tau(\partial U)=\inf\left\{t>0:B(t)\in\partial U\right\}$. Then the function $u:\overline{U}\to R$ given by

$$u(x)=E_x\left[\phi\left(B(\partial U)\right)\right],\ \text{for } x\in\overline{U}, \quad (11)$$

is the unique continuous function harmonic on U with $u(x)=\phi(x)$ for all $x\in\partial U$.

Proof. The function u is bounded and hence harmonic on U. It remains to show that the Poincare cone condition implies the boundary condition. Fix $z\in\partial U$, then there is a cone $C_z(\alpha)$ based at z with angle $\alpha>0$ with $C_z(\alpha)\cap B(z,h)\subset U^c$. By Lemma 1, for any positive integer k and $h^1>0$, we have

$$P_x\left\{\tau\left(\partial\beta\left(z,h^1\right)\right)<\tau\left(C_z(\alpha)\right)\right\}\le a^k \quad (12)$$

For all x with $|x-z|<2^{-k}h^1$. Given $\varepsilon>0$, there is a $0>\delta\le h$ such that $|\phi(y)-\phi(z)|<\varepsilon$ for all $y\in\partial U$ with $|y-z|<\delta$. For all $x\in\overline{U}$ with $|z-x|<2^{-k}\delta$,

$$|u(x)-u(z)|=\left|E_x\phi\left(B\left(\tau(\partial U)\right)\right)-\phi(z)\right|\le E_x\left|\phi\left(B\left(\tau(\partial U)\right)\right)-\phi(z)\right|. \quad (13)$$

If the Brownian motion hits the cone $C_z(\alpha)$, which is outside the domain U, before the sphere $\partial B(z,\delta)$, then $\left|z-B\left(\tau(\partial U)\right)\right|<\delta$, and $\phi\left(B\left(\tau(\partial U)\right)\right)$ is close to $\phi(z)$. The complement has small probability. More precisely, (13) is bounded above by

$$2\|\phi\|\infty P_x\left\{\tau\left(\partial B(z,\delta)\right)<\tau\left(C_z(\alpha)\right)\right\}+\varepsilon P_x\left\{(\partial U)<\tau\left(\partial B(z,\delta)\right)\right\}\le 2\|\phi\|\infty^k+\varepsilon. \quad (14)$$

This implies that u is continuous on $\overline{U}$.

Conclusively, If the Poincare cone condition holds at every boundary point, we simulate the solution of the Dirichlet problem by running many independent Brownian motions, starting in $x\in U$ until they hit the boundary of U and letting $u(x)$ be the average of the values of ϕ on the hitting points.

Example: Take a solution $v:B(0,1)\to R$ of the Dirichlet problem on the planar disc $B(0,1)$ with boundary condition $\phi:\partial B(0,1)-R$ Let $U=\left\{x\in R^2:0<|x|<1\right\}$ be the punctured disc. We claim that $u\left(x=E_x\left[\phi\left(B\left(\tau(\partial U)\right)\right)\right]\right)$ fails to solve the Dirichlet problem on U with boundary condition $\phi:\partial B(0,1)\cup\{0\}\to R$ if $\phi(0)\ne v(0)$. Indeed, as planar Brownian motion is outside the domain U, and the first hitting time τ of $\partial U=\partial B(0,1)\cup\{0\}$ agree almost surely with the first hitting time of $\partial B(0,1)$. Then, by Theorem 3.12, $u(0)=E_0\left[\phi\left(B(\tau)\right)\right]=v(0)\ne\phi(0)$.

We now show the techniques we have developed so far can be used to prove a classical result from harmonic analysis, Liouville's theorem, by probabilistic means. The proof uses the reflection principle for higher- dimensional Brownian motion.

Theorem 3: *[Liouville's theorem]* Any bounded harmonic function on R^d is constant[7].

Proof. Let $u:R^d \to [-M,M]$ be a harmonic function x,y two distinct points in R^d, and H the hyperplace so that the reflection in *H* takes x to y. Let $\{B(t):t\geq 0\}$ be Brownian motion started at x, and $\{\overline{B}(t):t\geq 0\}$ its reflection in *H*. Let $\tau(H)=\min\{t:B(t)\in H\}$ and note that

$$\{B(t):t\geq\tau(H)\}\underline{\underline{d}}\{\overline{B}(t):t\geq\tau(H)\}. \tag{15}$$

Harmonicity implies that $E_x\left[u\left(B(t)\right)\right]=u(x)$ and decomposing the above into $t<\tau(H)$ and $t\geq\tau(H)$ we get

$$u(x)=E_x\left[u\left(B(t)\right)1_{\{t<(H)\}}\right]+E_x\left[u\left(B(t)\right)1_{\{t\geq\tau(H)\}}\right]. \tag{16}$$

A similar equality holds for $u(y)$. Now, using (15),

$$\left|u(x)-u(y)\right|=E\left\|\left[u\left(B(t)\right)1_{\{t<\tau(H)\}}\right]-E\left[u\left(\overline{B}(t)\right)1_{\{t<\tau(H)\}}\right]\right\|\leq 2Mp\{t<\tau(H)\}\to 0, \quad \text{as } t\to\infty.$$ Thus

$u(x)=u(y)$, and since x and y were chosen arbitrarily, u must be constant.

Hence, the dirichlet problem for harmonic functions always has a solution, and that solution is unique, when the boundary is sufficiently smooth and f(s) is continuous. More precisely, it has a solution when $\partial D\in C^{(1,\alpha)}$ for 0 < α, where $C^{(1,\alpha)}$ denotes the **Holder condition[7].**

Maximum Principles

In convex optimization, the maximum principle state that the maximum of a convex function on a compact convex set is attained on the boundary [9].

Harmonic functions are the classical example to which the strong maximum principle applies. Formally, if *f* is a harmonic function, then *f* cannot exhibit a true local maximum within the domain of definition of *f*. in other words, either *f* is a constant function, or, for any point x_0 inside the domain of *f*, there exist other points arbitrarily close to x_0 at which *f* takes larger values [1].

Let *f* be defined on some connected open subset D of the Euclidean space R^n. if x_0 is a point in D such that $f(x_0)\geq f(x)$ for all x in a neighbourhood of x_0, then the function *f* is constant on D.

Harmonic functions satisfy the following maximum principles: if *R* is any compact subset of U, then *f*, restricted to *R*, attains its maximum and minimum on the boundary of *R*. If U is connected, this means that *f* cannot have local maxima or minima, other than the exceptional case where *f* is constant. Similar properties can be shown for subharmonic functions

Theorem 4: *(Maximum principle)* Suppose $u:R^d\to R$ is a function, which is subharmonic on an open connected set $U\subset R^d$.

(i) If u attains its maximum in U, then u is a constant.

(ii) If u is continuous on $\overline{U}$ and U, then u is bound and then,

$$\max_{x\in\overline{U}} u(x)=\max_{x\in\partial U} u(x)$$

Note that if u is harmonic, the theorem may be applied to both u and $-u$. Hence the conclusions of the theorem also hold with 'maximum' replaced by 'minimum'.

Proof. (i) Let M be the maximum. Note that $V=\{x\in U:u(x)=M\}$ is relatively closed in U. Since U is open, for any $x\in V$, and then is a ball $\beta(x,r)\subset U$. By the mean-value property of u,

$$M=u(x)\leq\frac{1}{L\left(\beta(x,r)\right)}\int_{\beta(x,r)}u(y)\,dy\leq M. \tag{17}$$

Equality holds everywhere, and as $u(y)\leq M$ for all $y\in\beta(x,r)$, we infer that $u(y)=M$ almost everywhere on $\beta(x,r)$. By continuity this implies $\beta(x,r)\subset V$. Hence V is also open, and by assumption nonempty. Since U is connected we get that $V=U$. Therefore, u is constant on U.

(ii) Since u is continuous and $\overline{U}$ is closed and bounded, u attains a maximum on $\overline{U}$. By (i) the maximum has to be attained on ∂U.

Alternatively, Suppose $u_1, u_2 : R^d \to R$ are functions, which are harmonic on a bounded domain $U \subset R^d$ and continuous on $\overline{U}$. If u_1 and u_2 agree on ∂U, then they are identical.

Proof. By Theorem 4 (ii) to u_1-u_2 we obtain that

$$\sup_{x\in\overline{U}}\left\{u_1(x)-u_2(x)\right\}=\sup_{x\in\partial U}\left\{u_1(x)-u_2(x)\right\}=0.$$

Hence $u_1(x) \le u_2(x)$ for all $x \in \overline{U}$. Applying the same argument to u_1-u_2, one sees that $\sup_{x\in\overline{U}}\left\{u_2(x)-u_1(x)\right\}=0$. Hence $u_1(x)=u_2(x)$ for all $x\in\overline{U}$.

We can now formulate the basic fact on which the relationship of Brownian motion and harmonic functions rests.

Theorem 5: Suppose U is a domain, $\left\{B(t): t\ge 0\right\}$ a Brownian motion started inside U and $\tau=\tau(\partial U)=\min\left\{t\ge 0: B(t)\in\partial U\right\}$ the first hitting time of its boundary. Let $\phi:\partial U\to R$ be measurable, and such that the function $u: U\to R$ with

$$u(x)=E_x\left[\phi\left(B(\tau)\right)1\left\{\tau<\infty\right\}\right], \tag{19}$$

for every $x\in U$, is locally bounded. Then, u is a harmonic function.

Proof. The proof uses only the strong Markov property of Brownian motion and the mean value characterization of harmonic functions. For a ball $B(x,\delta)\subset U$ let $\inf\left\{t>0: B(t)\notin B(x,\delta)\right\}$, then the strong Markov property implies that

$$u(x)=E_x\left[\phi\left(B(\tau)\right)1\left\{\tau<\infty\right\}\middle|F^+(\tilde{\tau})\right]\Big]=E_x\left[u\left(B(\tilde{\tau})\right)\right]=\int_{\partial B(x,r)}u(y)\varpi_{x,\delta}(dy),$$

Where $\varpi_{x,\delta}$ is the uniform distribution on the sphere $\partial B(x,\delta)$. Therefore, u has the mean value property and hence it is harmonic on U by Theorem 1.

Heuristics for the proof

The weak maximum principle for harmonic functions is a simple consequence of facts from calculus. The key ingredient for the proof is the fact that, by the definition of a harmonic function, the Laplacian of *f* is zero. Then, if x_0 is a non-degenerate critical point of *f(x)*, we must be seeing a saddle point, since otherwise there is no chance that the sum of the second derivatives of *f* is zero.

Conclusion

This analysis shows that the set of harmonic functions on a given open set U can be seen as the kernel of the Laplace operator ∇ and is therefore a vector space over R. Therefore, the application of sums, differences and scalar multiples of harmonic functions are again harmonic.

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