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The Concept of Convexity in Fuzzy Set Theory

[1] Rauf K. , [1]Omolehin J. O. , [2]Ma'ali A. I., [2]Badeggi A. Y. and [3]Lukuman A.

[1] Mathematics Department, University of Ilorin, Ilorin, Nigeria.
[2] Mathematics/Computer Science Department, IBB University, Lapai, Nigeria.
[3] Federal College of Fisheries and Marine Technology, Lagos State, Nigeria.

Abstract

The notions of convex analysis are indispensable in theoretical and applied Mathematics especially in the study of Calculus where it has a natural generalization for the several variables case.

This paper investigates the concept of Fuzzy set theory in relation to the idea of convexity. Some fundamental theorems were considered.

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1.0 Introduction

The sets whose elements have degrees of membership are called fuzzy sets. These sets were introduced in [1] as an extension of the classical notion of set.

In fuzzy logic, exact reasoning is viewed as a limiting case of approximate reasoning, everything is a matter of degree, knowledge is interpreted as a collection of elastic or fuzzy constraint on a collection of variables and inference is viewed as a process of propagation of elastic constraints. Indeed, Boolean logic is a subset of fuzzy logic because any logical system can be fuzzified.

These logics provide solutions to the problems of vagueness which departs from the all or nothing logic. They logically redefine yes or no ideas in proper form. [2, 3, 4, 5, 6].

1.1 Fuzzy set operation

1.1.1 Union

The membership function of the Union of two fuzzy sets A and B with membership functions μ_A and μ_B respectively is defined as the maximum of the two individual membership functions. This is called maximum criterion.

$$\mu_{A\cup B} = \max(\mu_A, \mu_B)$$

The Union operation in fuzzy set theory is the equivalent of the OR operation in Boolean algebra.

1.1.2 Intersection

The membership function of the intersection of two fuzzy sets A and B with membership functions μ_A and μ_B respectively is defined as the minimum of the two individual membership functions. This is called minimum criterion.

$$\mu_{A\cap B} = \min(\mu_A, \mu_B).$$

The intersection operation in Fuzzy set theory is the equivalent of the AND operations in Boolean algebra.

[1]Corresponding authors: ***Rauf K.,*** E-mail-, Tel. +2348094036003

1.1.3 Complement

The membership function of the complement of a fuzzy set A with membership function μ_A is defined as the negation of the specified membership function. This is called the negation criterion. $\mu_{\bar{A}} = 1 - \mu_A$

The complement operation in Fuzzy set theory is the equivalent of the NOT operation in Boolean algebra.
The following rules, which are common in Classical set theory, also apply to Fuzzy set theory: De Morgan's law, Associativity, Commutativity and Distributivity See Zadeh [1, 7]. Systems undergo three transformations to become system input viz: Fuzzification, Rulebase, and Defuzzification process.

1.1.4 Convex set:

Let K be a fuzzy set in n-dimensional Euclidian vector (linear) space, $\mathbb{R}^n$, then K is convex if for any two arbitrary points s, r ∈K the following conditions are satisfied.
(a) $r = (r_i : i \in \mathbb{N}_n), s = (s_i : i \in \mathbb{N}_n)$ for all positive integers ℕ.
(b) A point $t = \lambda r_1 + (1 - \lambda)s_1$ for all $i \in \mathbb{N}_n$, real number $\lambda \in [0,1]$. This implies that a
fuzzy set is convex if every point on the line segment connecting two points s and r in K is also a member of K.

1.1.5 Example

The membership function defined by

$$\mu_R = \begin{cases} [1 + (x-n)^{-2}]^{-1} & x > n \\ 0 & otherwise \end{cases}$$

is convex for all $n \in \mathbb{R}$.

1.1.6 Membership function of fuzzy set:

A function is said to be membership function of fuzzy set if each element of its domain is mapped to [0,1]. Consequently, the range must be in the interval [0,1].

1.1.7 Expression for Fuzzy set:

If every membership function μ_A in crisp set maps whole members in universal set X to {0,1} then the mapping is said to be mapping for fuzzy set, that is $\mu_A : X \to \{0, 1\}$.

1.1.8 α-cut of Fuzzy set:

α-cut of a fuzzy set is the set of all points $x \in X$ such that every membership function
$\mu(x) \geq \alpha$. See [8] and [9] for further information.

2. Formulation of Results

In this section, we shall consider some fundamental theorems that are useful in the study of convex fuzzy space.

Theorem 2.1

All the α-cuts of any fuzzy set K defined on $\mathbb{R}^n$ are convex if and only if
$\mu_A(\lambda r + (1 - \lambda)s) \geq Min[\mu_A(r),\ \mu_A(s)]$ for all $r, s \in \mathbb{R}^n, \lambda \in [0,1]$.

Proof:

Let K be any fuzzy set defined on $\mathbb{R}^n$ and all α-cuts be in K then for all $r, s \in \mathbb{R}^n, \lambda \in [0,1]$ and if $r < s$, then the membership function $\mu_A(\lambda r + (1-\lambda)s) \geq \mu_A(r)$. Similarly if $> s$ $\mu_A(\lambda r + (1-\lambda)s) \geq \mu_A(s)$. This implies that $\mu_A(\lambda r + (1-\lambda)s) \geq Min[\mu_A(r), \mu_A(s)]$ for all $r, s \in \mathbb{R}^n$.

Conversely, if $\mu_A(\lambda r + (1-\lambda)s) \geq Min[\mu_A(r), \mu_A(s)]$ then the crisp set in K whose elements have at least α degree of membership in fuzzy set K defined on $\mathbb{R}^n$ is convex, that is $K_\alpha = \{x \in X: \mu_A(x) \geq \alpha\}$, where X is the universal set that mapped to [0,1] by membership function $\mu_A(x)$. If such crisp sets in K defined on $\mathbb{R}^n$ exist then they are all convex for all $r, s \in \mathbb{R}^n$ and $\lambda \in [0,1]$. Hence the proof.

Theorem 2.2

Any convex fuzzy set in $\mathbb{R}^n$ of a convex fuzzy set in $\mathbb{R}^n$ is a convex fuzzy set in $\mathbb{R}^n$.

Proof

Let K be a convex fuzzy set in $\mathbb{R}^n$, then we need to show that R(K) is convex if R is a function of convex fuzzy set of K. Since K is convex, we have $t = \lambda r_i + (1-\lambda)k_i$ for $i \in \mathbb{N}, \lambda \in [0,1]$ and for any arbitrary points r_i and k_i in K. It follows that the function $\mu_k(t)$ is convex in $\mathbb{R}^n$. Consequently, the function $\mu_R(t)$ with R a function of K and mapping of one-one correspondent onto $[0, 1]$ is convex. This implies that $\mu_R(t) \geq Min[\mu_R(r_j),\ \mu_R(k_j)],\quad r_j,\ k_j \in R$.

Theorem 2.3

Let R and K be any convex fuzzy set in $\mathbb{R}^n$ then, the sum of R and K is a convex fuzzy set in $\mathbb{R}^n$.

Proof

Since R and K are convex fuzzy sets in $\mathbb{R}^n$ then the membership function of any two points from either R or K are also members (which are subsets) of the sum of R and K. Hence, any membership function in the sum of R and K must have a convex range in $\mathbb{R}^n$.

Theorem 2.4

Let α be a positive scalar and K be a convex fuzzy set in $\mathbb{R}^n$ then, αK is a convex fuzzy set in $\mathbb{R}^n$.

Proof

If K is a convex fuzzy set in $\mathbb{R}^n$ then, αK will translates or shrink the image domain depending on the value on positive scalar α, so that it becomes a convex fuzzy set in $\mathbb{R}^n$.

Theorem 2.5

Suppose that $f_1 \cdots f_n$ are non-negative convex fuzzy set with the same domain of definition. Then, $(f_1 \cdots f_n)^{\frac{1}{n}}$ is a convex fuzzy set in $\mathbb{R}^n$.

Proof

The proof is similar to the proof of Theorem 2.4.

3.0 Conclusion

The theory of fuzzy set has found much interest among physical scientists especially mathematicians and there are numerous new applications, extensions, refinements and variants of the theory.

In line with this, Theorem 2.1 is able to provide relationship (necessary and sufficient condition) between alpha-cuts and convexity condition while Theorems 2.2, 2.3, 2.4 and 2.5 are new properties to show that convex fuzzy set obeys some inclusion theorems and are closed in $\mathbb{R}^n$. These, to the knowledge of the authors, are new results in the literature.

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