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Stability Results for Multistep Iteration Satisfying a General Contractive Condition of Integral Type in a Normed Linear Space

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Abstract

In this paper, we prove some stability results for multistep iteration scheme by using maps satisfying contractive condition of integral type in a normed linear space. Our results are generalizations and extensions of some of the results of Berinde [1], Olatinwo [2], Osilike and Udomene [3], Rhoades [4, 5] and some other numerous results in the literature.

Keywords: Multistep iteration, contractive condition of integral type, normed linear spaces.

1.0 Introduction

Let E be a normed space and $T: E \to E$ a self map of E. For $x_0 \in E$, the Mann Iteration scheme [6] is the sequence $\{x_n\}_{n=0}^{\infty}$ given by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n, \tag{1.1}$$

where $\{\alpha_n\}_{n=0}^{\infty} \subset [0,1]$ such that $\sum_{n=0}^{\infty} \alpha_n \to \infty$.

For $x_0 \in E$, the sequence $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n$$
$$y_n = (1-\beta_n)x_n + \beta_n T x_n, \quad n \geq 0 \tag{1.2}$$

where $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are appropriate sequences in [0,1] such that $\sum_{n=0}^{\infty} \alpha_n \to \infty$ is called the Ishikawa iteration scheme [7].

For $x_0 \in E$, the sequence $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n$$
$$y_n = (1-\beta_n)x_n + \beta_n T z_n$$
$$z_n = (1-\gamma_n)x_n + \gamma_n T x_n, \qquad n \geq 0 \tag{1.3}$$

where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$ are appropriate sequences in [0,1] such that $\sum_{n=0}^{\infty} \alpha_n \to \infty$ is called the Noor iteration scheme [8].

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For $x_0 \in E$, the sequence $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y^1{}_n$$

$$y_n^i = (1-\beta_n^i)x_n + \beta_n^i T y_n^{i+1}, \; i = 1,2,3,\ldots,k-2$$

$$y_n^{k-1} = (1-\beta_n^{k-1})x_n + \beta_n^{k-1} T x_n, \qquad k \geq 2 \tag{1.4}$$

where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n^i\}_{n=0}^{\infty}$, $i = 1,2,,\ldots,k-1$ are appropriate sequences in [0,1] such that $\sum_{n=0}^{\infty} \alpha_n \to \infty$ is called a multistep iteration scheme [9].

Definition 1.1. Let (X,d) be a complete metric space and $T : X \to X$ a selfmap of X. Suppose that $F_T = \{p \in X : T_p = p\}$ is the set of fixed points of T. Let $\{x_n\}_{n=0}^{\infty} \subset X$ be the sequence generated by an iteration scheme involving T which is defined by

$$x_{n+1} = f(T, x_n), \; n = 0,1,2,\ldots \tag{1.5}$$

where $x_0 \in X$ is the initial approximation and f is some function. Suppose $\{x_n\}_{n=0}^{\infty}$ converges to a fixed point p of T. Let $\{y_n\}_{n=0}^{\infty} \subset X$ and set $\varepsilon_n = d(y_{n+1}, f(T, y_n)),\; n = 0,1,2,\ldots$. Then the iteration scheme (1.5) is T-stable if and only if $\lim_{n\to\infty} y_n = p.$

If in (1.5), $f(T, x_n) = Tx_n$, $n = 0,1,2,\ldots$, then we have the Picard iteration scheme, while we obtain the Mann iteration scheme if

$$f(T, x_n) = (1-\alpha_n)x_n + \alpha_n T x_n,\; n = 0,1,2,\ldots, \alpha_n \in [0,1]. \tag{1.6}$$

Finally, if $f(T, x_n) = (1-\alpha_n)x_n + \alpha_n T y^1{}_n$

$$y_n^i = (1-\beta_n^i)x_n + \beta_n^i T y_n^{i+1}, \; i = 1,2,3,\ldots,k-2$$

$$y_n^{k-1} = (1-\beta_n^{k-1})x_n + \beta_n^{k-1} T x_n, \qquad k \geq 2, \tag{1.7}$$

we obtain the multistep iteration process.

Definition 1.2 [2]. Let (X,d) be a complete metric space and $T : X \to X$ a selfmap of X, there exists $a \in [0,1)$ and a monotone increasing function $\varphi : R^+ \to R^+$ with $\varphi(0) = 0$ such that

$$d(Tx, Ty) \leq ad(x, y) + \varphi(d(x, Tx)), \; \forall \; x, y \in X. \tag{1.8}$$

Motivated by the contractive condition given by Olatinwo [2], Branciari [10] and Rhoades [11] gave the following contractive definition:

Definition 1.3 [2]. For a selfmapping $T : X \to X$, there exists a real number $c \in [0,1)$ and a monotone increasing functions $\nu, \psi : R^+ \to R^+$ such that $\psi(0) = 0$ and $\forall \; x, y \in X$, we have

$$\int_0^{d(Tx,Ty)} \varphi(t)d\nu(t) \le \psi\left(\int_0^{d(x,Tx)} \varphi(t)d\nu(t)\right) + c\int_0^{d(x,y)} \varphi(t)d\nu(t) \qquad (1.9)$$

where $\varphi : R^+ \to R^+$ is a Lebesgue-Stieltjes integrable mapping which is summable, nonnegative and such that for each

$$\varepsilon > 0, \int_0^{\varepsilon} \varphi(t)d\nu(t) > 0. \qquad (1.10)$$

Remark 1.4. If in (1.10), $\varphi(t) = 1$ and $\nu(t) = t$, we have condition (1.8).

Lemma 1.5 [1]. If δ is a real number such that $0 \le \delta < 1$, and $\{\varepsilon_n^1\}_{n=0}^{\infty}$ is a sequence of positive numbers such that $\lim_{n\to\infty} \varepsilon_n^1 > 0$, then for any sequence of positive numbers $\{u_n\}_{n=0}^{\infty}$ satisfying

$$u_{n+1} \le \delta u_n + \varepsilon_n^1, \; n = 0,1,2,..., \qquad (1.11)$$

we have $\lim_{n\to\infty} u_n = 0.$

Lemma 1.6 [2]. Let (X, d) be a complete metric space and $\varphi : R^+ \to R^+$ a Lebesgue Stieltjes integrable mapping which is summable, nonnegative and such that for each $\varepsilon > 0, \int_0^{\varepsilon} \varphi(t)d\nu(t) > 0.$ Suppose that $\{u_n\}_{n=0,}^{\infty}$ $\{v_n\}_{n=0}^{\infty} \subset X$ and $\{a_n\}_{n=0}^{\infty} \subset (0,1)$ are sequences such that

$$\left| d(u_n, v_n) - \int_0^{d(u_n,v_n)} \varphi(t)d\nu(t) \right| \le a_n, \qquad (1.12)$$

with $\lim_{n\to\infty} a_n = 0.$ Then

$$d(u_n, v_n) - a_n \le \int_0^{d(u_n,v_n)} \varphi(t)d\nu(t) \le d(u_n, v_n) + a_n. \qquad (1.13)$$

Proof:

By letting $y = d(u_n, v_n) - \int_0^{d(u_n,v_n)} \varphi(t)d\nu(t)$ and using the definition of modulus function in $|y|$, we have (1.13).

Remark 1.7 [2]. Lemma 1.6 is also applicable in normed linear space since metric is induced by a norm. that is

$d(x, y) = \| x - y \|, \; \forall \; x, y \in X$, whenever we are working in a normed linear space.

Several authors have obtained different stability results for various maps in literature. It has been shown that the recent stability results of Olatinwo [2] generalizes and extends some other known results like that of Berinde [1], Osilike and Udomene [3], Rhoades [4, 5] and Harder and Hicks [12].

The main aim of this paper is to establish the stability of multistep iteration schemes (1.4) satisfying a general contractive condition of integral type in a normed linear space, thereby generalizing and extending the results of Olatinwo [8], which is in turn a generalization of other known stability results in the literature.

2 Main Results

Theorem 2.1. Let $(E,\|.\|)$ be a normed space and $T:E\to E$ a self mapping of E satisfying

$$\int_0^{d(Tx,Ty)}\varphi(t)d\nu(t)\le\psi\left(\int_0^{d(x,Tx)}\varphi(t)d\nu(t)\right)+c\int_0^{d(x,y)}\varphi(t)d\nu(t), \tag{2.1}$$

where $c\in[0,1)$ and ν,ψ are monotonic increasing functions defined by $\nu,\psi:R^+\to R^+$ such that $\psi(0)=0$, $\forall$ $x,y\in E$ and $\varphi:R^+\to R^+$ is a Lebesgue-Stieltjes integrable mapping which is summable, nonnegative and such that for each $\varepsilon>0$, $\int_0^{\varepsilon}\varphi(t)d\nu(t)>0.$ Suppose T has a fixed point $p.$ For $x_0\in E,$ Let $\{x_n\}_{n=0}^{\infty}$ be the multistep iteration schemes defined by (1.4), where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{i}$ are sequences in [0,1] such that $0<\beta^1<\beta_n^{\ 1}$ $(n=0,1,2,...).$ Then the multistep iteration schemes (1.4) is $T-$stable.

Proof:

Let $\{z_n\}_{n=0}^{\infty}$, $\{u_n^{\ i}\}_{n=0}^{\infty}$ for $i=1,2,...,k-1$ be real sequences in $E.$ Suppose $\varepsilon_n=\|z_{n+1}-(1-\alpha_n)z_n-\alpha_nTw_n^{\ 1}\|,$
$n=0,1,2,...$ (2.2)

where

$$w_n^{\ 1}=(1-\beta_n^{\ i})z_n+\beta_n^{\ i}Tw_n^{\ i+1},\quad i=1,2,...,k-2$$

$$w_n^{\ k-1}=(1-\beta_n^{\ k-1})z_n+\beta_n^{\ k-1}Tz_n,\ \ k\ge 2, \tag{2.3}$$

and let $\lim_{n\to\infty}\varepsilon_n=0,$ then we shall show that $\lim_{n\to\infty}z_n=p.$

Let $\{a_n\}_{n=0}^{\infty}\subset(0,1),$ then by Lemma (1.6), we have

$$\int_0^{\|z_{n+1}-p\|}\varphi(t)d\nu(t)\le\|z_{n+1}-p\|+a_n$$

$$\le(\|z_{n+1}-(1-\alpha_n)z_n-\alpha_nTw_n^{\ 1}\|-a_n)+(1-\alpha_n)(\|z_n-p\|-a_n)$$
$$+\alpha_n(\|Tp-Tw_n^{\ 1}\|-a_n)+3a_n \tag{2.4}$$

$$\le\int_0^{\varepsilon_n}\varphi(t)d\nu(t)+(1-\alpha_n)\int_0^{\|z_n-p\|}\varphi(t)d\nu(t)+\psi(\int_0^{\|p=Tp\|}\varphi(t)d\nu(t))$$
$$+c\int_0^{\|p-w_n^{\ 1}\|}\varphi(t)d\nu(t)+3a_n. \tag{2.5}$$

$$\| w_n^{\ 1} - p \| \le (1-\beta_n^{\ 1}) \| z_n - p \| + \beta_n^{\ 1} \| Tw_n^{\ 2} - p \|$$

$$\le (1-(1-c)\beta_n^{\ 1} - (1-c)c\beta_n^{\ 1}\beta_n^{\ 2} - (1-c)c^2\beta_n^{\ 1}\beta_n^{\ 2}\beta_n^{\ 3}$$

$$- \ldots - (1-c)c^{k-2}\beta_n^{\ 1}\beta_n^{\ 2}\beta_n^{\ 3}\ldots\beta_n^{\ k-1}) \| z_n - p \|$$

$$\le (1-(1-c)\beta_n^{\ 1} \| z_n - p \| . \tag{2.6}$$

Substituting (2.6) in (2.5), we obtain

$$\int_0^{\|z_{n+1}-p\|} \varphi(t)d\nu(t) \le \int_0^{\varepsilon_n} \varphi(t)d\nu(t) + (1-\alpha_n) \int_0^{\|z_n-p\|} \varphi(t)d\nu(t)$$

$$+ c\alpha_n(1-(1-c)\beta_n^{\ 1}) \int_0^{\|z_n-p\|} \varphi(t)d\nu(t) + 3a_n. \tag{2.7}$$

$$\int_0^{\|z_{n+1}-p\|} \varphi(t)d\nu(t) \le (1-(1-c)\alpha_n - (1-c)c\alpha_n\beta_n^{\ 1}) \int_0^{\|z_n-p\|} \varphi(t)d\nu(t) + \int_0^{\varepsilon_n} \varphi(t)d\nu(t) + 3a_n. \tag{2.8}$$

Since $0 < \alpha < \alpha_n$ and $0 < \beta^1 < \beta_n^{\ 1}$ for $n \ge 1,$ we have

$$\int_0^{\|z_n-p\|} \varphi(t)d\nu(t) \le (1-(1-c)\alpha - (1-c)c\alpha\beta^1) \int_0^{\|z_n-p\|} \varphi(t)d\nu(t) + \int_0^{\varepsilon_n} \varphi(t)d\nu(t) + 3a_n. \tag{2.9}$$

Using Lemma 1.5, where

$$0 \le \delta = 1-(1-c)\alpha - (1-c)c\alpha\beta^1 < 1, \tag{2.10}$$

$$u_n = \int_0^{\|z_n-p\|} \varphi(t)d\nu(t) \tag{2.11}$$

and $$\varepsilon_n^{\ 1} = \int_0^{\varepsilon_n} \varphi(t)d\nu(t) + 3a_n, \tag{2.12}$$

with $$\lim_{n\to\infty} \varepsilon_n^{\ 1} = \lim_{n\to\infty} \left(\int_0^{\varepsilon_n} \varphi(t)d\nu(t) + 3a_n \right) = 0. \tag{2.13}$$

and the fact that $\int_0^{\varepsilon} \varphi(t)d\nu(t) > 0,$ for each $\varepsilon > 0$ by Lemma 1.6, we have

$\lim_{n\to\infty} \int_0^{\|z_n-p\|} \varphi(t)d\nu(t) = 0.$ This implies that $\lim_{n\to\infty} \| z_n - p \| = 0.$ That is $\lim_{n\to\infty} z_n = p.$

Conversely, let $\lim_{n\to\infty} z_n = p,$ by the contractive condition in Theorem 2.1, Lemma 1.6 and triangular inequality, we have

$$\int_0^{\varepsilon_n}\varphi(t)d\nu(t) = \int_0^{\|z_{n+1}-p+p-(1-\alpha_n)z_n-\alpha_n Tw_n^1\|}\varphi(t)d\nu(t) \le \int_0^{\|z_{n+1}-p\|}\varphi(t)d\nu(t) + \int_0^{\|p-(1-\alpha_n)z_n-\alpha_n Tw_n^1\|}\varphi(t)d\nu(t)$$

$$\le \int_0^{\|z_{n+1}-p\|}\varphi(t)d\nu(t) + (1-(1-c)\alpha_n-(1-c)c\alpha_n\beta_n^{\ 1})\int_0^{\|z_n-p\|}\varphi(t)d\nu(t) + 3a_n$$

$$\le \int_0^{\|z_{n+1}-p\|}\varphi(t)d\nu(t) + (1-(1-c)\alpha-(1-c)c\alpha\beta^1)\int_0^{\|z_n-p\|}\varphi(t)d\nu(t) + 3a_n. \quad (2.14)$$

Since by assumption, $\lim_{n\to\infty} z_n = p$ implies $\lim_{n\to\infty 0}\int_0^{\|z_{n+1}-p\|}\varphi(t)d\nu(t) = 0$ and that

$1-(1-c)\alpha-(1-c)c\alpha\beta^1 < 1,$ it follows that

$\int_0^{\varepsilon}\varphi(t)d\nu(t) = 0$ implies $\lim_{n\to\infty}\varepsilon_n = 0.$ This completes the proof.

Theorem 2.1 leads to the following Corollaries:

Corollary 2.2. Let $(E,\|.\|)$ be a normed space and $T: E \to E$ a self mapping of E satisfying

$\int_0^{d(Tx,Ty)}\varphi(t)d\nu(t) \le \psi\left(\int_0^{d(x,Tx)}\varphi(t)d\nu(t)\right) + c\int_0^{d(x,y)}\varphi(t)d\nu(t),$ where $c\in[0,1)$ and ν,ψ are monotonic increasing functions defined by $\nu,\psi: R^+ \to R^+$ such that $\psi(0)=0$, $\forall\ x,y\in E$ and $\varphi: R^+\to R^+$ is a Lebesgue-Stieltjes integrable mapping which is summable, nonnegative and such that for each $\varepsilon>0,$ $\int_0^{\varepsilon}\varphi(t)d\nu(t)>0.$ Suppose T has a fixed point $p.$ For $x_0\in E,$ Let $\{x_n\}_{n=0}^{\infty}$ be the Noor iteration schemes defined by (1.3), where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$, are sequences in [0,1] such that and $0<\alpha<\alpha_n$ $(n=0,1,2,...).$ Then the Noor iteration scheme (1.3) is $T-$ stable.

Corollary 2.3. Let $(E,\|.\|)$ be a normed space and $T: E \to E$ a self mapping of E satisfying

$\int_0^{d(Tx,Ty)}\varphi(t)d\nu(t) \le \psi\left(\int_0^{d(x,Tx)}\varphi(t)d\nu(t)\right) + c\int_0^{d(x,y)}\varphi(t)d\nu(t),$ where $c\in[0,1)$ and ν,ψ are monotonic increasing functions defined by $\nu,\psi: R^+ \to R^+$ such that $\psi(0)=0$, $\forall\ x,y\in E$ and $\varphi: R^+\to R^+$ is a Lebesgue-Stieltjes integrable mapping which is summable, nonnegative and such that for each $\varepsilon>0,$ $\int_0^{\varepsilon}\varphi(t)d\nu(t)>0.$ Suppose T has a fixed point $p.$ For $x_0\in E,$ Let $\{x_n\}_{n=0}^{\infty}$ be the Ishikawa iteration scheme defined by (1.2), where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ are sequences in [0,1] such that and $0 < \alpha < \alpha_n \quad (n = 0,1,2,...)$. Then the Ishikawa iteration scheme (1.2) is $T-$stable.

Corollary 2.4. Let $(E, \|\,.\,\|)$ be a normed space and $T : E \to E$ a self mapping of E satisfying

$$\int_0^{d(Tx,Ty)} \varphi(t) d\nu(t) \le \psi\left(\int_0^{d(x,Tx)} \varphi(t) d\nu(t) \right) + c \int_0^{d(x,y)} \varphi(t) d\nu(t),$$ where $c \in [0,1)$ and ν, ψ are monotonic increasing functions defined by $\nu, \psi : R^+ \to R^+$ such that $\psi(0) = 0$, $\forall \; x, y \in E$ and $\varphi : R^+ \to R^+$ is a Lebesgue-Stieltjes integrable mapping which is summable, nonnegative and such that for each $\varepsilon > 0, \int_0^{\varepsilon} \varphi(t) d\nu(t) > 0.$ Suppose T has a fixed point p. For $x_0 \in E$, Let $\{x_n\}_{n=0}^{\infty}$ be the Mann iteration schemes defined by (1.1), where $\{\alpha_n\}_{n=0}^{\infty}$ is sequence in [0,1] such that and $0 < \alpha < \alpha_n \quad (n = 0,1,2,...)$. Then the Mann iteration scheme (1.1) is $T-$stable.

Corollary 2.5. Let $(E, \|\,.\,\|)$ be a normed space and $T : E \to E$ a self mapping of E satisfying

$$\int_0^{d(Tx,Ty)} \varphi(t) d\nu(t) \le \psi\left(\int_0^{d(x,Tx)} \varphi(t) d\nu(t) \right) + c \int_0^{d(x,y)} \varphi(t) d\nu(t),$$ where $c \in [0,1)$ and ν, ψ are monotonic increasing functions defined by $\nu, \psi : R^+ \to R^+$ such that $\psi(0) = 0$, $\forall \; x, y \in E$ and $\varphi : R^+ \to R^+$ is a Lebesgue-Stieltjes integrable mapping which is summable, nonnegative and such that for each $\varepsilon > 0, \int_0^{\varepsilon} \varphi(t) d\nu(t) > 0.$ Suppose T has a fixed point p. For $x_0 \in E$, Let $\{x_n\}_{n=0}^{\infty}$ be the Picard iteration scheme defined by (1.5). Then the Picard iteration scheme (1.5) is $T-$stable.

Conclusion 2.6. Theorem 2.1 is a generalization and extension of Theorem 3 of Berinde [1],Theorem 3.1 and Theorem 3.2 of Olatinwo [2], Theorem 24 of Rhoades [4], Theorem 2 of Rhoades [5] and Theorem 3 of Harder and Hicks [12] in the sense that it considered the multistep iteration schemes which is more general than the Noor, Ishikawa, Mann and Picard iteration schemes considered by various other authors.

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