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A Note on Using Unbounded Functions on Totally Bounded Sets in the Context of the Lebesgue Covering Lemma

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Abstract

From a real-valued function f, unbounded on a totally bounded subset S of a metric space, we construct a Cauchy sequence in S on which f is unbounded. Taking f to be a reciprocal Lebesgue number function, for an open cover of S, gives a rapid proof that S is compact when it is complete, without recourse to sequential compactness or the Lebesgue covering lemma. Finally, we apply the same reasoning to another function f to give sequential compactness.

Keywords: Pseudometric space, Cauchy sequence, total boundedness, Lebesgue number, Lebesgue covering lemma.

1.0 Introduction

An examination of both well established accounts of compactness in metric spaces, for example [1] and [2], and more recent work in this area [3], led the author to explore whether the simplicity and full power of the Lebesgue number of an open cover was being fully exploited when showing that a totally bounded complete space is compact.

Classically, the Lebesgue number is defined for the open cover $\mathcal{C}$ and its existence is deduced with the Lebesgue Covering Lemma. This leads to somewhat complicated subsequence arguments in conjunction with variously selected open balls, see [3] and [2].

Our approach is to define a reciprocal Lesbesgue number function $L_{\mathcal{C}}$ at each point being covered, so that that establishing the existence of the Lebesgue number reduces to deciding whether this function is bounded. A simple construction shows that any unbounded function on a subset of a totally bounded space is unbounded on a Cauchy sequence. This combines with special properties of $L_{\mathcal{C}}$ to avoid use of sequential compactness altogether in showing that a totally bounded complete subset of a metric space is compact. Replacing $L_{\mathcal{C}}$ by another function M_T for $T \subset S$, allows entirely similar reasoning for sequential compactness.

Examples of recent work using total boundedness in the context of abstract uniform spaces are, particularly for topological groups, [4] and even rings [5]. We have explored how the concepts and results of this paper generalise to uniform spaces, but the obvious generalisation requires the existence of a countable fundamental system of entourages for the uniformity. But then it is well known [6] that the uniformity is induced by a pseudometric, so the work is presented for a pseudometric space X with pseudometric d, for which the defining conditions are, for all $x, y, z \in X$:

$d(x,y) \geq 0$ and $d(x,x) = 0$; (non-negative, semi-definite)
$d(x,y) = d(y,x)$; (symmetric)
$d(x,z) \leq d(x,y) + d(y,z)$. (triangle inequality)

For each $x \in X$ and $r > 0$, the open balls with centre x and radius r and $r/2$ are denoted

$$B(x,r) = \{y \in X \mid d(x,y) < r\} \text{ and } B_2(x,r) = \{y \in X \mid d(x,y) < r/2\}.$$

Note two simple consequences of the triangle inequality,

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$$y \in B_2(x,r) \text{ implies} B_2(y,r) \subset B(x,r), \tag{1}$$

and

$$y,z \in B_2(x,r) \text{implies} y \in B(z,r). \tag{2}$$

2.0 Totally Bounded Subsets

Most authors work only with totally bounded spaces. When applying the concept to a subset S of space X, there is a choice as to whether to use open balls whose centres are in S or in X. We make the first choice:

2.1 Definition *A subset S of X is called totally bounded in X if given $\varepsilon > 0$ there is a finite cover of S by open balls $B(x,\varepsilon)$ with centres in $x \in S$.*

But, with this definition, it takes several steps to show that a non-empty subset T of a totally bounded subset S in X is again totally bounded in X. Indeed, given $\varepsilon > 0$, take a finite open cover $\{B_2(s_1,\varepsilon),\dots,B_2(s_n,\varepsilon)\}$ of S with centres in S: for each i with $1 \le i \le n$, take some $t_i \in T \cap B_2(s_i,\varepsilon)$ when this set is non-empty, and take t_i to be any point of T otherwise. Then, using implication (1),$\{B(t_1,\varepsilon),\dots,B(t_n,\varepsilon)\}$ is an open cover of T by open balls with centres in T.

3.0 The Construction

3.1 Theorem*Let S_0 be a totally bounded subset of a metric space X, and suppose that $f: S_0 \to \boldsymbol{R}$ is not bounded above. Then there is a Cauchy sequence $a_1, a_2, \dots$ of points of S_0 such that $f(a_n) > n$ for all $n \in \boldsymbol{N}$.*

Proof. We recursively define a sequence $a_1, a_2, \dots$ of points of S_0 and a descending chain $S_0 \supset S_1 \supset \cdots$ of subsets of X so that, for each $n \in \mathbf{N}$,

$$f \text{isunboundedaboveon} S_n = B\left(a_n, \frac{1}{n}\right) \cap S_{n-1} \text{ and } f(a_n) > n. \tag{3}$$

Now suppose for a non-negative integer n that a subset of S_n of S_0 has been defined such that, if $n > 0$, condition (3) is satisfied, whilst if $n = 0$, S_n is simply S_0. Because $S_n \subset S_0$, it is totally bounded, and there is a finite open cover $\{B_2(x_1, 1/(n+1)),\dots,B_2(x_m, 1/(n+1))\}$ of S_n. Since f is unbounded on S_n, it is also unbounded on the points of S_n lying within at least one of these balls $B_2(b_{n+1}, 1/(n+1))$, say. Thus there is a point $a_{n+1} \in S_n \cap B_2(b_{n+1}, 1/(n+1))$ for which $f(a_{n+1}) > n+1$, and by implication (1), since $B_2(b_{n+1}, 1/(n+1)) \subset B(a_{n+1}, 1/(n+1))$, taking $S_{n+1} = S_n \cap B(a_{n+1}, 1/(n+1))$ satisfies condition (3) with $n+1$ in place of n, and the recursion can continue indefinitely.

Now for any $m,n > 2K$, we have $a_m, a_n \in S_{2K} \subset B_2(a_{2K}, 1/K)$, so that, by implication (1),

$$a_m \in B(a_n, 1/K)$$

and thus $a_1, a_2, \dots$ is a Cauchy sequence with the required property.

3.2 Corollary *Let $f: S \to \boldsymbol{R}$ be a function locally bounded above on a complete totally bounded subset S of X. Then f is bounded above on S.*

Proof. If f is not bounded above, by the theorem, there is a Cauchy sequence $a_1, a_2, \dots$ of S, with $f(a_n) > n$ for each $n \in \mathbf{N}$. Because S is complete, this sequence converges to some $a \in S$, and because f is locally bounded at a, there is some $r > 0$ and some $M \in \mathbf{R}$such that

$$f(x) \le M \text{ forall} x \in B(a,r) \cap S.$$

But since, for all $n > M$, we have $f(a_n) > M$ and thus $a_n \notin B(a,r)$, this gives contradiction. Thus f is bounded on S as required.

4.0 Application to Open Covers and Compactness

4.1 Definition *Let $\mathcal{C}$ be a collection of open subsets of a metric space X, and let $V = \bigcup_{U\in\mathcal{C}} U$. For any $x \in V$, we have $x \in U$ for some $U \in \mathcal{C}$, and thus, for suitable large $n \in \boldsymbol{N}$ we have $B(x, 1/n) \subset U$, because U is open. Thus the **reciprocal Lebesgue number function** $L_{\mathcal{C}}: V \to \boldsymbol{N}$ given by*

$$L_{\mathcal{C}}(x) = \inf\{n \in \boldsymbol{N} \mid B\left(x, \frac{1}{n}\right) \subset U \; for \; some \; U \in \mathcal{C}\}$$

is well defined.

A crucial property of this function is its local boundedness as follows.

4.2 Proposition *In the notation of 4.1, for any $x \in V$, taking $\delta = 1/L_{\mathcal{C}}(x)$, we see that the function $L_{\mathcal{C}}$ is defined on $B_2(x, \delta)$ and*

$$y \in B_2(x, \delta) \; implies \; L_{\mathcal{C}}(y) \leq 2L_{\mathcal{C}}(x). \qquad (4)$$

Proof. Taking $n \geq L_U(x)$, there is some $U \in \mathcal{C}$ such that $B(x, 1/n) \subset U$. However, by implication (1), for $y \in B_2(x, 1/n)$, we have

$$B_2(y, 1/n) \subset B(x, 1/n) \subset U$$

and thus $L_{\mathcal{C}}(y) \leq 2n$. Putting $n = L_{\mathcal{C}}(x)$ gives property (4).

Equally important is the following classical result.

4.3 Lemma *If $\mathcal{C}$ is an open cover of a totally bounded subset S of a metric space X and $L_{\mathcal{C}}$ is bounded on S, then there is finite subcover of $\mathcal{C}$ for S.*

Proof. Suppose $L_{\mathcal{C}}(x) \leq K \in \mathbf{N}$ for all $x \in S$ and let $B(x_1, 1/K), \dots, B(x_n, 1/K)$ be an open cover for S with centres in S. Since, for each i there is $U_i \in \mathcal{C}$ with $B(x_i, 1/K) \subset U_i$, the collection $\{U_1, \dots, U_n\}$ is an open cover for S.

4.4 Theorem *Any complete totally bounded subset of a pseudometric space X is compact.*

Proof. Take an open cover $\mathcal{C}$ of a complete totally bounded subset S of X. Because $L_{\mathcal{C}}$ is locally bounded on S, corollary 3.2 shows that it is bounded on S, and hence lemma 4.3 shows that there is a finite subcover of $\mathcal{C}$ for S.

5.0 Application to Infinite Subsets and Sequential Compactness

Suppose $T \subset S \subset X$ and that T has no limit point in S. Then fixing $x \in S$, there is $n \in \mathbf{N}$ such that $B(x, 1/n) \cap T$ is finite and we can define

$$M_T(x) = \inf\{n \in \mathbf{N} | B\left(x, \frac{1}{n}\right) \cap T \text{is finite}\}.$$

Using implication (1) we have

$$y \in B_2\left(x, \frac{1}{M_T(x)}\right) \text{implies} B_2(y, 1/M_T(x)) \subset B(x, 1/M_T(x))$$

and thus $M_T(y) \leq 2M_T(x)$ for $y \in B_2(x, 1/M_T(x))$, so that M_T is locally bounded on S.

5.1 Theorem *Let T be a subset of a complete totally bounded subset S of a pseudometric space X. If T has no limit point in S, then T is finite.*

Proof. Using corollary 3.2, we see that $M_T(x) < K$ for all $x \in S$, for some suitable $K \in \mathbf{N}$. But S has a finite cover $\{B(x_1, 1/K), \dots, B(x_n, 1/K)\}$ for some points $x_1, \dots, x_n$ of S. Since each $M_T(x_i) < K$, the sets $B(x_i, 1/K) \cap T$ are finite for $1 \leq i \leq m$, and thus T is finite.

An immediate consequence of this theorem is sequential compactness of S, see for example [2].

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