Journal of the Nigerian Association of Mathematical Physics
Volume **19 (November, 2011), pp 619 – 622**

Application of New Variational Homotopy Perturbation Method to Painlevé Equation I

[1]Taiwo O. A. and [2]Abolarin O. E.
[1]Department of Mathematics, University of Ilorin, Ilorin, Nigeria.
[2]Department of Mathematics and Statistics, The Federal Polytechnic, P.M.B. 55, Bida, Nigeria.

Abstract

In this study, New Variational Homotopy Pez'turbation Method (NVHPM) was employed to approximate solutions of Painlevé Equation I, with its initial conditions. NVHPM based on coupling of VIM and HPM as introduced by Hesameddini and Latifizadeh.[1}, with some good modification was engaged for the purpose of this study. In this paper we apply the NVHPM to Painlevé Equation I as given by Hesameddini and Peyrovi[2] to find its approximate solutions. The advantage of the new Scheme is that it does not require discritization, linearization or any restrictive assumption of any form before it is applied. Numerical comparisons are made between VIM/HPM and NVHPM results.

Keywords: Painlevé Equations, Variational Iteration Method, Homotopy Perturbation Method, New Variational Homotopy Perturbation Method, Ordinary Differential Equations.

1.0 Introduction

The Painlevé Equations and thier solutions arises in parts of pure and applied mathematics and theoretical physics. Painlevé considered a wide class of second order equations and classified them to the nature of singularities. Painlevd and his coworkers found essentially six different equations within the class considered whose solutions are singled valued as functions of complex independent variables, except possibly at the fixed singularities of the coefficients. These are known as Painlevé transcendents and have a great variety of interesting properties and applications. Many reseachers have investigated Painlevé equations using several techniques. In this work we are going to study Painlevé equation I as also studied by Hesameddini and Peyrovi [2] which is well known as

$$u'' = 6u^2 + x \qquad (1.1)$$

with the initial conditions

$$u(0) = 0, u'(0) = 1 \qquad (1.2)$$

by using the New Variational Homotopy Perturbation Method (NVHPM). This method can be applied successfully to various types of ordinary and partial differential equations.

2 Basic Idea of Variational Iteration Method

To illustrate the basic ideas of variational iteration method according to He [3-7], we consider the following differential equation:

$$LU+NU=g(t) \qquad (2.1)$$

where:

L is a linear operator
N is a nonlinear operator
g(t) is the homogenous term

According to variational iteration method, we can construct a correction functional as follows:

(2.2)

where A is a general Lagrangian multiplier which can be identified optimally via the variational theory. The subscript 'n' indicates the nth approximation and $\bar{U}_n$ is considered as a restricted variation i.e. $\delta\bar{U}_n = 0$. The solution of the linear problems can be solved in a single iteration step due to the exact identification of the Lagrange multiplier.

Corresponding author: ***Abolarin O. E.,*** E-mail: sola4one@yahoo.com, Tel. -

The successive approximation $u_{n+1,} n \geq 0$ of the solution U will be readily obtained upon using the determined Lagrange multiplier and any selective function U_o, consequently, the solution is given by $U = \lim_n \to \infty\ U_n$.

3 Basic Idea of Homotopy Perturbation Method

Linear and Non-linear Phenomena are of fundamental importance in various fields of science and engineering. Most models of real-life problems are still very difficult to solve. Therefore, an approximate analytical solution such as Homotopy Perturbation Method was introduced by He [8]. To explain this method, we consider the following general non-linear differential equation:

$$A(u) - f(r) = 0,\ r \in \Omega \tag{3.1}$$

with the boundary conditions

$$B\left(u, \delta u/\delta n\right) = 0, r \in \Gamma \tag{3.2}$$

where A,B, f(r) and r are a general differential operator, a boundary operator, a known analytical function and the boundary of the domain Ω, respectively.

The operator A can, generally speaking, be divided into parts L and N (say), where L is linear part, while N is non-linear part. Equation (3.1) therefore can be rewritten as follows:

$$L(u) + N(u) - f(r) = 0 \tag{3.3}$$

By the Homotopy techniques, we constructed a Homotopy

$v(r,p)$: $\Omega \times [0,1] \to R$ which satisfies:

$$H(v,p) = (1-p)\,[L(v)-L(u_o)] + p[A(v)-f(r)] = 0, \tag{3.4}$$
$$p \in \left[0, 1\right],\ r \in \Omega$$

or

$$H(v,p) = L(v)-L(u_o)+pL(u_o)+p[N(v)-f(r)] = 0 \tag{3.5}$$

Where p $\in$ [0,1], is an embedding parameter, while u_o is an initial approximation of equation (3.1), which satisfies the boundary condition. Obviously, from equation (3.4) and (3.5) we will have:

$$H(v,0) = L(v)-L(u_o) = 0 \tag{3.6}$$
$$H(v, 1) = A(v)-f(r) = 0, \tag{3.7}$$

the changing process of p from zero to unity is just that of H(v, p) from L(v)-L(u_o) to A(v)-f(r). In topology, this is called deformation, L(v)-L(u_o) and A(v)-f(r) are called homotopic.

According to the Homotopy Perturbation method, we can first use the embedding parameter p as a small parameter, and assume that the solution of equations (3.4) and (3.5) can be written as a power series in p:

$$V = v_0 + pv_1 + p^2v_2 + p_3v_3 + \tag{3.8}$$
$$U = \lim_{p\to 1} V = v_o + v_2 + v_3 + \tag{3.9}$$

The combination of the perturbation method and the Homotopy method is called the Homotopy Perturbation Method, which eliminates the drawbacks of the traditional perturbation methods while keeping all its advantages.

The series (3.9) is convergent for most cases. However, the convergent rate depends on the non-linear operator. He [8] made the following suggestions:

The second derivative of N(v) with respect to v must be small because the parameter may be relatively large, i.e. p $\to$ 1. The norm of $L^{-}1\delta N/\delta v$ must be smaller than one so that the series converges.

The introduction of equation (3.8) into equations (3.4) and (3.5) and comparison of like powers of p gives solution of various orders.

4 New Variational Homotopy Perturbation Method

To illustrate the basic concept of the New Variational Homotopy Perturbation Method, we consider the following general differential equation:

$$LU + NU = g(t) \tag{4.1}$$

where L is a linear operator, N a non-linear operator, and g(t) is the homogenous term. By the variational iteration method, we construct a correction functional

$$u_{n+1}\left(t\right) = u_n\left(t\right) + \int_0^1 \lambda\left[LU_n\left(\tau\right) + N\bar{U}_n\left(\tau\right) - g\left(\tau\right)\right]d\tau \tag{4.2}$$

where λ is a Lagrange multiplier according to Barari et al [9], which can be identified optimally via variational theory. The subscript n denotes the n-th approximation, $\bar{U}_n$ is considered as a restricted variational, that is, $\delta\bar{U}_n = 0$; and equation (4.2) is called a correction functional.

Now we apply the Homotopy Perturbation Method discussed earlier to the correction functional in equation (4.2) and (3.9) into the correction functional, we have the following:

$$U_0 + PU_1 + P^2U_2 + \cdots = U_0 + P\int_0^t \lambda(\tau)L\Big[\big(U_0 + PU_1 + P^2U_2 + \ldots\big)$$

$$+\big(N\overline{U}_0 + PN\overline{U}_1 + P^2\overline{U}_2 + \ldots\big)\Big]d\tau - \int_0^t \lambda(\tau)g(\tau)d\tau \tag{4.3}$$

This can be expressed as:

$$\sum_{n=0}^{\infty} P^nU_n = U_0(t) - \int \lambda(\tau)P\Big[\sum_{n=0}^{\infty} P^nLU_n + \sum_{n=0}^{\infty} P^nN\overline{U}_n\Big]d\tau$$

$$-\int_0^t \lambda(\tau)\,g(\tau)d\tau \tag{4.4}$$

Hence, equation (4.4) represents the coupling of variational iteration and Homotopy Perturbation methods. The comparison of the coefficients of like powers of P gives solutions of various orders, this implies:

$$P^0 : U_0 = U_0(t) - \int_0^t \lambda(\tau)g(\tau)d\tau$$

$$P^1 : U_1 = \int_0^t \lambda(\tau)\big(LU_0 + N\overline{U}_0\big)d\tau$$

$$P^2 : U_2 = \int_1^t \lambda(\tau)\big(LU_1 + N\overline{U}_1\big)d\tau \tag{4.5}$$

.

.

.

$$P^n : U_n = \int_{n-1}^t \lambda(\tau)\big(LU_{n-1} + N\overline{U}_{n-1}\big)d\tau$$

Therefore, the series solution is given as:

$$U(t) = U_0 + U_1 + U_2 + \ldots\ldots\ldots\ldots + U_n \tag{4.6}$$

Hence,

$$U(t) = U_0 + \int_0^t \lambda(\tau)\big(LU_0 + N\overline{U}_0\big)d\tau + \int_1^t \lambda(\tau)\big(LU_1 + N\overline{U}_1\big)d\tau +$$

$$\ldots\ldots\ldots + \int_{n-1}^t \lambda(\tau)\big(LU_{n-1} + N\overline{U}_{n-1}\big)d\tau - \int_0^t \big(\lambda(\tau)g(\tau)\big)d\tau \tag{4.7}$$

5. NVHPM for Painlevé Equation

Using NVHPM for equation (1.1) given by Hesameddini and Peyrovi [2], accordingly to VIM we first construct a correction functional as follows:

$$u_{n+1}(x) = u_n(x) + \int_0^x (\xi - x)\left[\frac{d^2}{d\xi^2}u_n(\xi) - 6\overline{u}_n^2(\xi) - \xi\right]d\xi \tag{5.1}$$

Applying NVHPM to equation (5.1), we have:

$$u_0 + pu_1 + p^2u_2 + .. = u_0 + p\int_0^x (\xi - x)\left[\frac{d^2}{d\xi^2}\big(u_0 + pu_1 + p^2u_2 + ..\big)\right.$$

$$\left.-\,6\,\big(\overline{u}_0^2 + p\overline{u}_1^2 + p^2\overline{u}_2^2 + ..\big) - \xi\right]d\xi \tag{5.2}$$

Comparing the coefficients of like powers of p, we have:

$$p^0 : u_0(x) = x + \frac{x^3}{6}$$

$$p^1 : u_1 = \frac{x^3}{6} + \frac{24x^6}{12} + \frac{6x^8}{18}$$

A tabular comparison between our solution via NVHPM and that of VIM/HPM given by Hesameddini and Peyrovi [2] is presented below.

Table 1: comparison between HPM/VIM and NVHPM results

X	HPM/VIM	NVHOM	ERROR
0.0100	0.01000017167	0.01000036333	1.9166×10^{-7}
0.0200	0.02000141333	0.02000314680	1.73347×10^{-6}
0.0300	0.03000490505	0.03001143146	6.5264×10^{-6}
0.0400	0.04001194694	0.04002902152	1.7075×10^{-5}
0.0500	0.05002395937	0.05006044793	3.6489×10^{-5}
0.0600	0.06004248311	0.06011097337	6.8490×10^{-5}
0.0700	0.07006917951	0.07018659882	1.1742×10^{-4}
0.0800	0.08010583081	0.08029407152	1.8824×10^{-4}
0.0900	0.09015434044	0.09044089431	2.8655×10^{-4}

6 Conclusion

In this paper the NVHPM was applied to finding the approximate solutions of Painlevé Equation I with initial conditions. The numerical solutions are compared with the numerical solutions of HPM/VIM in table 1. the results showed that the new variational homotopy perturbation method is in close agreement with that of HPM/VIM and we will achieve to a desired approximation by means of few iterations. However, it is on note that in this paper we showed application of NVHPM for a problem which has no exact solutions.

References

[1] E. Hesameddini and H. Latifizadeh , "New Approach to Mixture of the Adomian Decomposition Homotopy Perturbation for Approximate and Analytical Solution of Integral Equations Fractional Differential Equations", Applied Mathematical Sciences, vol.3, (2009) ,no.572797- 2809.

[2] E. Hesameddini and A. Peyrovi , "The use of Variational Iteration Method and Homotopy Perturbation Method for Painlevé Equation I" ,Applied Mathematical Sciences, vol.3,no.38, (2009), 1861-1871.

[3] J.H. He , "Variational Iteration MethodA Kind Of Nonlinear Analytical Technique: Some Examples", International Journal of Non-Linear Mechanics, vol. 34, no. 4,(1999), pp. 699-708.

[4] J.H. He ,"Variational Iteration Method For Autonomous Ordinary Differential Systems", Applied Mathematics and Computation, vol. 114, no. 2-3,(2000), pp. 115-123.

[5] J.H. He, "Some Asymptotic Methods For Strongly Nonlinear Equations", International Journal of Modern Physics B, vol. 20, no. 10,(2006), pp. 1141-1199.

[6] J.H. He, "Variational Iteration Method-Some Recent Results And New Interpretations", Journal of Computational and Applied Mathematics, vol. 207, no. 1, (2007), pp. 3-17.

[7] J.H. He, "The Variational Iteration Method For Eighth-Order InitialBoundary Value Problems", Physica Scripta, vol. 76, no. 6,(2007), pp. 680-682.

[8] J.H. He ," A Coupling Method Of A Homotopy Technique And A Perturbation Technique For Nonlinear Problems", International Journal of Non-Linear Mechanics, vol. 35, no. 1,(2000), pp. 37-43.

[9] A.Barari, A.R. Ghotbi and D.D.Ganji, "Variational Iteration Method and Homotopy-perturbation Method for Solving Different Types of Wave Equations" ,Journal of Applied Science 8(1),(2008):120-126.