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Behavioural Pattern of Invertibility Parameter of Arima Model

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Abstract

In this paper, we present an inverted form of Autoregressive Integrated Moving Average processes (ARIMA) of various orders. Investigation was carried out on the behavioural pattern of invertibility parameter π_i of the ARIMA (p, d, q) for various p and d. It was deduced that behaviour of invertibility parameter π_i depends on the order of autoregressive part (p), the order of integrated part (d), positive and negative values of moving average parameter (θ).

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1.0 Introduction

When fitting ARIMA model, we must check whether stationarity and invertibility conditions are satisfied. All ARIMA(0,d,q) models are stationary. But we must check that they satisfy another condition, invertibility. All ARIMA(*p, d*, 0) models are invertible, but depending on the values of the parameters they may not be stationary. Therefore while a model has various representations it makes sense to look for the simplest representations to estimate. In considering estimates of parameters of time series model, [1] computed exact log likelihood of time series model, proposed and justified an asymptotic approximation of it.

Granger and Andersen [2] proposed a generalized definition of invertibility and applied it to linear, non-linear and bilinear models. It was shown that some recently studied non-linear models are not invertible, but conditions for invertibility can be achieved for the other models.

Anderson [3] deduced conditions for the general Moving Average process, of order *q*, to be invertible or borderline non-invertible. He termed the conditions as acceptability conditions. It turned out that they depended on the magnitude of the final moving average parameter, θ_q. If $\left|\theta_q\right| > 1$, the process is not acceptable. Should $\left|\theta_q\right| = 1$, the conditions, for any particular *q*, follow simply - if use is made of the remainder theorem. When $\left|\theta_q\right| < 1$, an appeal was made to ROUCH* E'S theorem, to establish the conditions. Analogous stationarity results immediately follow for autoregressive processes.

Mikosch et al [4] considered ARMA process of the form $\phi(L)X_t = \theta(L)Z_t$, where the innovations Z_t belong to the domain of attraction of a stable law, so that neither the X_t have a finite variance. They estimated the coefficients of ϕ and θ using Whittle estimator based on sample periodogram of X sequence. They showed that their estimators were consistent, obtained their asymptotic distributions and showed that they converged to the true values faster than in the usual $\mathfrak{L}^2$ case.

Triacca [5] analyzed the relationship between feedback in stochastic systems, Granger causality and a measure of dissimilarity between ARMA models in bivariate vector autoregression. In particular, he considered a bivariate vector autoregressive processes of order *p* (a bivariate VAR(*p*) process) and argued that if the distance between the univariate ARMA models implied by the VAR representation is greater than a certain number that is a function of *p*, then Granger causality must exist in at least one direction in the variables.

In this work, ARIMA model of various orders are presented in inverted form with a view of examining the behavioural pattern of invertibility parameter of ARIMA (*p, d, q*) for various integer value of *p* and *d*.

Autoregressive Moving Average Process

The need for estimating the parameters of an ARMA (*p*, *q*) process arises in many applications both in signal processing and in automatic control. One subset of ARMA models are the so-called autoregressive, or AR models while the other is moving average or MA models. The notation ARMA (p, q) refers to the model with p autoregressive terms and q moving average terms. This model contains the AR (p) and MA (q) models,

$$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \ldots + \phi_p y_{t-p} + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2} + \ldots + \theta_q \varepsilon_{t-q} \quad (1)$$

The terms $\phi_1 y_{t-1}$ through $\phi_p y_{t-p}$ are the autoregressive portion of the filter. The terms ε_t through $\theta_q \varepsilon_{t-q}$ are a moving average of the white noise input process.

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Specification of ARMA models in terms of lag operator

When the models are specified in terms of the lag operator L, the AR (p) model is given by [6] as

$$\varepsilon_t = \left(1 - \sum_{i=1}^{p} \phi_i L^i\right) y_t = \phi(L) y_t, \text{ where } \phi(L) = 1 - \sum_{i=1}^{p} \phi_i L^i$$

and MA (q) model is given by $y_t = \left(1 + \sum_{i=1}^{p} \theta_i L^i\right)\varepsilon_t = \theta(L)\varepsilon_t$, where $\theta(L) = 1 + \sum_{i=1}^{p} \theta_i L^i$. ARMA (p, q) is given as

$$\left(1 - \sum_{i=1}^{p} \phi_i L^i\right) y_t = \left(1 + \sum_{i=1}^{p} \theta_i L^i\right)\varepsilon_t \qquad (2)$$

or more concisely, $\phi(L) y_t = \theta(L)\varepsilon_t$ which implies $y_t = \psi(L)\varepsilon_t$, where

$$\pi(L) = \frac{\phi(L)}{\theta(L)} = \frac{1 - \phi_1 L - \phi_2 L^2 - \ldots - \phi_p L^p}{1 + \theta_1 L + \theta_2 L^2 + \ldots + \theta_p L^p} \qquad (3)$$

An ARIMA process y_t is invertible (strictly, an invertible function of ε_t) if there is a

$$\pi(L) = \pi_0 + \pi_1 L + \pi_2 L^2 + \_\_\_$$

with $\sum_{j=0}^{\infty} |\pi_j| < \infty$ and $\varepsilon_t = \pi(L) y_t$.

Autoregressive Integrated Moving Average (ARIMA) Model

This is a generalisation of ARMA model. It consists of Autoregressive model (AR), Integrated part (I; which is differencing term) and moving average model (MA). It is employed when time series is non stationary. *p* is the order of AR, *d* is the number of times a series is differenced to assume stationarity and *q* is the order of MA.

Invertibility of Arima Processes

An ARIMA (*p*, *d*, *q*) process is said to be invertible if the series converges in mean to ε_t as $p \to \infty$. This happens when $\theta(Z) = 0$ lie outside the unit circle. An ARIMA (*p*, *d*, *q*) process is invertible if the absolute value of the parameters of ARIMA (*p*, *d*, *q*) model satisfy $|\theta_i| < 1$ for $i = 1, \ldots, q$. The invertibility condition is satisfied if the absolute value of the parameters of ARMA (*p*, *q*) model satisfy $|\theta_i| < 1$ for $i = 1, \ldots, q$. If the roots of the polynomial $\theta(L)$ lie outside the unit circle, the ARMA (*p*, *q*) process is invertible and thus forecast can be done in as much it satisfies the invertibility condition. Let y_t be an ARMA process defined by $\phi(L) y_t = \theta(L)\varepsilon_t$. If $\phi(z) \neq 0$for all$|z| = 0$, then there are polynomials $\tilde{\phi}$ and $\tilde{\theta}$ and a white noise sequence ε_t such that y_t satisfies $\tilde{\phi}(L) y_t = \tilde{\theta}(L)\varepsilon_t$, and this is a causal, invertible ARMA process.

Presentation of some ARIMA processes inverted form

Here we present some ARIMA processes of various values of p and d in inverted form.

ARIMA (1,0,1)

Recall from equation (1), $p = 1$ and $q = 1$ gives

$$y_t = c + \phi y_{t-1} + \varepsilon_t + \theta \varepsilon_{t-1}$$

$$(1 + \theta L)\varepsilon_t = -c + y_t - \phi y_{t-1}$$

$$\varepsilon_t = -(1+\theta)^{-1} c + y_t - (\theta + \phi) y_{t-1} + (\theta^2 + \phi\theta) y_{t-2} - (\theta^3 + \phi\theta^2) y_{t-3} + \ldots$$

$$\varepsilon_t = -\frac{c}{(1+\theta)} + y_t + \sum_{i=1}^{\infty} (-1)^i (\theta^i + \phi\theta^{i-1}) y_{t-i} \qquad (4)$$

$$\varepsilon_t = -\frac{c}{(1+\theta)} + \sum_{i=0}^{\infty} \pi_i y_{t-i} \qquad (5)$$

where

$$\pi_i = \begin{cases} 1, & i = 0 \\ (-1)^i \left[\phi\theta^{i-1} + \theta^i\right], & i = 1, 2, 3, \ldots \end{cases} \qquad (6)$$

Equation (5) holds if $\theta \neq -1$.

ARIMA (1, 1, 1)

$$\Delta y_t = c + \phi \Delta y_{t-1} + \varepsilon_t + \theta \varepsilon_{t-1} \quad (7)$$
$$(1 + \theta L)\varepsilon_t = y_t - (1 + \phi) y_{t-1} + \phi y_{t-2} - c$$

$$\varepsilon_t = -(1+\theta)^{-1} c + y_t - (\theta + \phi + 1) y_{t-1} + (\theta^2 + \phi\theta + \theta + \phi) y_{t-2} - (\theta^3 + \phi\theta^2 + \theta^2 + \phi\theta) y_{t-3} + (\theta^4 + \phi\theta^3 + \theta^3 + \phi\theta^2) y_{t-4} - \ \dots \quad (8)$$

$$\varepsilon_t = -\frac{c}{(1+\theta)} + y_t - (\theta + \phi + 1) y_{t-1} + \sum_{i=2}^{\infty} (-1)^i [(\theta + \phi)(\theta^{i-1} + \theta^{i-2})] y_{t-i} \quad (9)$$

$$\varepsilon_t = -\frac{c}{(1+\theta)} + \sum_{i=0}^{\infty} \pi_i y_{t-i} \quad (10)$$

where

$$\pi_i = \begin{cases} 1, & i = 0 \\ -(\theta + \phi + 1), & i = 1 \\ (-1)^i [(\theta + \phi)(\theta^{i-1} + \theta^{i-2})], & i = 2, 3, 4, \dots \end{cases} \quad (11)$$

Equation (10) holds if $\theta \neq -1$.

ARIMA (1, 2, 1)

$$\Delta^2 y_t = c + \phi \Delta^2 y_{t-1} + \varepsilon_t + \theta \varepsilon_{t-1} \quad (12)$$
$$y_t - 2y_{t-1} + y_{t-2} = c + \phi(y_{t-1} - 2y_{t-2} + y_{t-3}) + \varepsilon_t + \theta \varepsilon_{t-1}$$

$$(1 + \theta L)\varepsilon_t = y_t - (2 + \phi) y_{t-1} + (1 + 2\phi) y_{t-2} - \phi y_{t-3} - c$$

$$\varepsilon_t = -(1 + \theta L)^{-1} c + (1 + \theta L)^{-1} [y_t - (2 + \phi) y_{t-1} + (1 + 2\phi) y_{t-2} - \phi y_{t-3}]$$

$$\varepsilon_t = -(1+\theta)^{-1} c + y_t - (\theta + \phi + 2) y_{t-1} + [\theta^2 + \theta(\phi + 2) + (2\phi + 1)] y_{t-2} - [\theta^3 + \theta^2(\phi + 2) + \theta(2\phi + 1) + \phi] y_{t-3} + [\theta^4 + \theta^3(\phi + 2) + \theta^2(2\phi + 1) + \theta\phi] y_{t-4} - [\theta^5 + \theta^4(\phi + 2) + \theta^3(2\phi + 1) + \theta^2\phi] y_{t-5} + \ \dots \quad (13)$$

$$\varepsilon_t = -\frac{c}{(1+\theta)} + y_t - (\theta + \phi + 2) y_{t-1} + [\theta^2 + \theta(\phi + 2) + (2\phi + 1)] y_{t-2} + \sum_{i=3}^{\infty} (-1)^i [\theta^i + \theta^{i-1}(\phi + 2) + \theta^{i-2}(2\phi + 1) + \theta^{i-3}\phi] y_{t-i} \quad (14)$$

$$\varepsilon_t = -\frac{c}{(1+\theta)} + \sum_{i=0}^{\infty} \pi_i y_{t-i} \quad (15)$$

where

$$\pi_i = \begin{cases} 1, & i = 0 \\ -(\theta + \phi + 2), & i = 1 \\ \theta^2 + \theta(\phi + 2) + (2\phi + 1), & i = 2 \\ (-1)^i [\theta^i + \theta^{i-1}(\phi + 2) + \theta^{i-2}(2\phi + 1) + \theta^{i-3}\phi], & i = 3, 4, 5, \dots \end{cases} \quad (16)$$

Equation (15) holds if $\theta \neq -1$.

ARIMA (1, 3, 1)

$$\Delta^3 y_t = c + \phi \Delta^3 y_{t-1} + \varepsilon_t + \theta \varepsilon_{t-1} \quad (17)$$
$$y_t - 3y_{t-1} + 3y_{t-2} - y_{t-3} = c + \phi(y_{t-1} - 3y_{t-2} + 3y_{t-3} - y_{t-4}) + \varepsilon_t + \theta \varepsilon_{t-1}$$

$$(1 + \theta L)\varepsilon_t = y_t - (3 + \phi) y_{t-1} + (3 + 3\phi) y_{t-2} - (1 + 3\phi) y_{t-3} + \phi y_{t-4} - c$$

$$\varepsilon_t = -(1+\theta L)^{-1}c + (1+\theta L)^{-1}[y_t - (3+\phi)y_{t-1} + (3+3\phi)y_{t-2} - (1+3\phi)y_{t-3} + \phi y_{t-4}]$$

$$\begin{aligned}\varepsilon_t = -(1+\theta)^{-1}c + y_t - (\theta+\phi+3)y_{t-1} &+ [\theta^2 + \theta(\phi+3) + (3+3\phi)]y_{t-2} \\ &- [\theta^3 + \theta^2(\phi+3) + \theta(3+3\phi) + (1+3\phi)]y_{t-3} \\ &+ [\theta^4 + \theta^3(\phi+3) + \theta^2(3+3\phi) + \theta(1+3\phi) + \phi]y_{t-4} \\ &- [\theta^5 + \theta^4(\phi+3) + \theta^3(3+3\phi) + \theta^2(1+3\phi) + \theta\phi]y_{t-5} \\ &+ [\theta^6 + \theta^5(\phi+3) + \theta^4(3+3\phi) + \theta^3(1+3\phi) + \theta^2\phi]y_{t-5} - \ \ldots \qquad (18)\end{aligned}$$

$$\begin{aligned}\varepsilon_t = -\frac{c}{(1+\theta)} + y_t - (\theta+\phi+3)y_{t-1} &+ [\theta^2 + \theta(\phi+3) + (3+3\phi)]y_{t-2} \\ &- [\theta^3 + \theta^2(\phi+3) + \theta(3+3\phi) + (1+3\phi)]y_{t-3} \\ &+ \sum_{i=4}^{\infty}(-1)^i[\theta^i + \theta^{i-1}(\phi+3) + \theta^{i-2}(3\phi+3) + \theta^{i-3}(1+3\phi) \\ &+ \theta^{i-4}\phi]y_{t-i} \qquad (19)\end{aligned}$$

$$\varepsilon_t = -\frac{c}{(1+\theta)} + \sum_{i=0}^{\infty}\pi_i y_{t-i} \qquad (20)$$

where

$$\pi_i = \begin{cases} 1, & i = 0 \\ -(\theta+\phi+3), & i = 1 \\ \theta^2 + \theta(\phi+3) + (3+3\phi), & i = 2 \\ -[\theta^3 + \theta^2(\phi+3) + \theta(3+3\phi) + (1+3\phi)], & i = 3 \\ (-1)^i[\theta^i + \theta^{i-1}(\phi+3) + \theta^{i-2}(3\phi+3) & \\ \qquad +\theta^{i-3}(1+3\phi) + \theta^{i-4}\phi], & i = 4, 5, 6, \ldots \end{cases} \qquad (21)$$

Equation (20) holds if $\theta \neq -1$.

ARIMA (2, 0, 1)

$$y_t = c + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \varepsilon_t + \theta\varepsilon_{t-1} \qquad (22)$$

$$(1+\theta L)\varepsilon_t = -c + y_t - \phi_1 y_{t-1} - \phi_2 y_{t-2}$$

$$\varepsilon_t = -(1+\theta L)^{-1}c + (1+\theta L)^{-1}(y_t - \phi_1 y_{t-1} - \phi_2 y_{t-2})$$

$$\varepsilon_t = -(1+\theta)^{-1}c + y_t - (\theta+\phi_1)y_{t-1} + (\theta^2 + \phi_1\theta - \phi_2)y_{t-2} - (\theta^3 + \phi_1\theta^2 - \phi_2\theta)y_{t-3} + (\theta^4 + \phi_1\theta^3 - \phi_2\theta^2)y_{t-4} - \ \ldots$$

$$\varepsilon_t = -\frac{c}{(1+\theta)} + y_t - (\theta+\phi_1)y_{t-1} + \textstyle\sum_{i=2}^{\infty}(-1)^i[\theta^i + \phi_1\theta^{i-1} - \phi_2\theta^{i-2}]y_{t-i} \qquad (23)$$

$$\varepsilon_t = -\frac{c}{(1+\theta)} + \sum_{i=0}^{\infty}\pi_i y_{t-i} \qquad (24)$$

where

$$\pi_i = \begin{cases} 1, & i = o \\ -(\theta+\phi_1), & i = 1 \\ (-1)^i[\theta^i + \phi_1\theta^{i-1} - \phi_2\theta^{i-2}], & i = 2, 3, 4, \ldots \end{cases} \qquad (25)$$

Equation (24) holds if $\theta \neq -1$. This result confines [7].

ARIMA (2, 1, 1)

$$\Delta y_t = c + \phi_1 \Delta y_{t-1} + \phi_2 \Delta y_{t-2} + \varepsilon_t + \theta \varepsilon_{t-1} \tag{26}$$

$$(1+\theta L)\varepsilon_t = y_t - (1+\phi_1)y_{t-1} + (\phi_1 - \phi_2)y_{t-2} + \phi_2 y_{t-3} - c$$

$$\varepsilon_t = -\frac{c}{(1+\theta)} + y_t - (\theta + \phi_1 + 1)y_{t-1} + (\theta^2 + \phi_1\theta + \theta + \phi_1 - \phi_2)y_{t-2} + \sum_{i=3}^{\infty}(-1)^i[\theta^i + \theta^{i-1}(1+\phi_1) + \theta^{i-2}(\phi_1 - \phi_2) - \phi_2\theta^{i-3}]y_{t-i} \tag{27}$$

$$\varepsilon_t = -\frac{c}{(1+\theta)} + \sum_{i=0}^{\infty}\pi_i y_{t-i} \tag{28}$$

where

$$\pi_i = \begin{cases} 1, & i = 0 \\ -(\theta + \phi_1 + 1), & i = 1 \\ \theta^2 + \phi_1\theta + \theta + \phi_1 - \phi_2, & i = 2 \\ (-1)^i[\theta^i + \theta^{i-1}(1+\phi_1) + \theta^{i-2}(\phi_1 - \phi_2) - \phi_2\theta^{i-3}], & i = 3, 4, \ldots \end{cases} \tag{29}$$

Equation (28) holds if $\theta \neq -1$.

ARIMA (2, 2, 1)

$$\Delta^2 y_t = c + \phi_1 \Delta^2 y_{t-1} + \phi_2 \Delta^2 y_{t-2} + \varepsilon_t + \theta \varepsilon_{t-1} \tag{30}$$

$$y_t - 2y_{t-1} + y_{t-2} = c + \phi_1(y_{t-1} - 2y_{t-2} + y_{t-3}) + \phi_2(y_{t-2} - 2y_{t-3} + y_{t-4}) + \varepsilon_t + \theta\varepsilon_{t-1}$$

$$(1+\theta L)\varepsilon_t = y_t - (2+\phi_1)y_{t-1} + (1 + 2\phi_1 - \phi_2)y_{t-2} - (\phi_1 - 2\phi_2)y_{t-3} - \phi_2 y_{t-4} - c$$

$$\varepsilon_t = -(1+\theta L)^{-1}c + (1+\theta L)^{-1}y_t - (2+\phi_1)(1+\theta L)^{-1}y_{t-1} + (1+2\phi_1-\phi_2)(1+\theta L)^{-1}y_{t-2} - (\phi_1 - 2\phi_2)(1+\theta L)^{-1}y_{t-3} - \phi_2(1+\theta L)^{-1}y_{t-4}$$

$$\varepsilon_t = -\frac{c}{(1+\theta)} + y_t - (\theta + \phi_1 + 2)y_{t-1} + [\theta^2 + \theta(2+\phi_1) + (1 + 2\phi_1 - \phi_2)]y_{t-2} - [\theta^3 + \theta^2(2+\phi_1) + \theta(1+2\phi_1-\phi_2) + (\phi_1 - 2\phi_2)]y_{t-3} + \sum_{i=4}^{\infty}(-1)^i[\theta^i + \theta^{i-1}(\phi+2) + \theta^{i-2}(1+2\phi_1-\phi_2) + \theta^{i-3}(\phi_1 - 2\phi_2) - \theta^{i-4}\phi_2]y_{t-i} \tag{31}$$

$$\varepsilon_t = -\frac{c}{(1+\theta)} + \sum_{i=0}^{\infty}\pi_i y_{t-i} \tag{32}$$

where

$$\pi_i = \begin{cases} 1, & i = 0 \\ -(\theta + \phi_1 + 2), & i = 1 \\ \theta^2 + \theta(2+\phi_1) + (1+2\phi_1-\phi_2), & i = 2 \\ -[\theta^3 + \theta^2(2+\phi_1) + \theta(1+2\phi_1-\phi_2) + (\phi_1 - 2\phi_2)], & i = 3 \\ (-1)^i[\theta^i + \theta^{i-1}(\phi_1+2) + \theta^{i-2}(1+2\phi_1-\phi_2) \\ \quad +\theta^{i-3}(\phi_1 - 2\phi_2) - \theta^{i-4}\phi_2], & i = 4, 5, 6, \ldots \end{cases} \tag{33}$$

Equation (32) holds if $\theta \neq -1$.

ARIMA (2, 3, 1)

$$\Delta^3 y_t = c + \phi_1 \Delta^3 y_{t-1} + \phi_2 \Delta^3 y_{t-2} + \varepsilon_t + \theta\varepsilon_{t-1} \tag{34}$$

$$y_t - 3y_{t-1} + 3y_{t-2} - y_{t-3} = c + \phi_1(y_{t-1} - 3y_{t-2} + 3y_{t-3} - y_{t-4}) + \phi_2(y_{t-2} - 3y_{t-3} + 3y_{t-4} - y_{t-5}) + \varepsilon_t + \theta\varepsilon_{t-1}$$

$$(1+\theta L)\varepsilon_t = y_t - (3+\phi_1)y_{t-1} + (3+3\phi_1-\phi_2)y_{t-2} - (1+3\phi_1-3\phi_2)y_{t-3} + (\phi_1-3\phi_2)y_{t-4} + \phi_2 y_{t-5} - c$$

$$\begin{aligned}\varepsilon_t = &-(1+\theta L)^{-1}c \\ &+ (1+\theta L)^{-1}[y_t - (3+\phi_1)y_{t-1} + (3+3\phi_1-\phi_2)y_{t-2} - (1+3\phi_1-3\phi_2)y_{t-3} + (\phi_1-3\phi_2)y_{t-4} \\ &+ \phi_2 y_{t-5}]\end{aligned}$$

$$\begin{aligned}\varepsilon_t = -\frac{c}{(1+\theta)} &+ y_t - (\theta+\phi_1+3)y_{t-1} + [\theta^2 + \theta(3+\phi_1) + (3+3\phi_1-\phi_2)]y_{t-2} \\ &- [\theta^3 + \theta^2(3+\phi_1) + \theta(3+3\phi_1-\phi_2) + (1+3\phi_1-3\phi_2)]y_{t-3} \\ &+ [\theta^4 + \theta^3(3+\phi_1) + \theta^2(3+3\phi_1-\phi_2) + \theta(1+3\phi_1-3\phi_2) + (\phi_1-3\phi_2)]y_{t-4} \\ &+ \sum_{i=5}^{\infty}(-1)^i[\theta^i + \theta^{i-1}(\phi_1+3) + \theta^{i-2}(3+3\phi_1-\phi_2) + \theta^{i-3}(1+3\phi_1-3\phi_2) + \theta^{i-4}(\phi_1-3\phi_2) \\ &- \theta^{i-5}\phi_2]y_{t-i}\end{aligned} \tag{35}$$

$$\varepsilon_t = -\frac{c}{(1+\theta)} + \sum_{i=0}^{\infty}\pi_i y_{t-i} \tag{36}$$

where

$$\pi_i = \begin{cases} 1, & i=0 \\ -(\theta+\phi_1+3), & i=1 \\ \theta^2 + \theta(3+\phi_1) + (3+3\phi_1-\phi_2), & i=2 \\ -[\theta^3 + \theta^2(3+\phi_1) + \theta(3+3\phi_1-\phi_2) + (1+3\phi_1-3\phi_2)], & i=3 \\ \theta^4 + \theta^3(3+\phi_1) + \theta^2(3+3\phi_1-\phi_2) \\ \qquad +\theta(1+3\phi_1-3\phi_2) + (\phi_1-3\phi_2), & i=4 \\ (-1)^i[\theta^i + \theta^{i-1}(\phi_1+3) + \theta^{i-2}(3+3\phi_1-\phi_2) \\ \quad + \theta^{i-3}(1+3\phi_1-3\phi_2) + \theta^{i-4}(\phi_1-3\phi_2) - \theta^{i-5}\phi_2], & i=5,6,7,\dots \end{cases} \tag{37}$$

Equation (37) holds if $\theta \neq -1$.

ARIMA (3, 0, 1)

$$y_t = c + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \phi_3 y_{t-3} + \varepsilon_t + \theta\varepsilon_{t-1} \tag{38}$$

$$(1+\theta L)\varepsilon_t = -c + y_t - \phi_1 y_{t-1} - \phi_2 y_{t-2} - \phi_3 y_{t-3}$$

$$\varepsilon_t = -(1+\theta L)^{-1}c + (1+\theta L)^{-1}(y_t - \phi_1 y_{t-1} - \phi_2 y_{t-2} - \phi_3 y_{t-3})$$

$$\begin{aligned}\varepsilon_t = -\frac{c}{(1+\theta)} &+ y_t - (\theta+\phi_1)y_{t-1} + (\theta^2 + \phi_1\theta - \phi_2)y_{t-2} \\ &+ \sum_{i=3}^{\infty}(-1)^i[\theta^i + \phi_1\theta^{i-1} - \phi_2\theta^{i-2} + \phi_3\theta^{i-3}]y_{t-i}\end{aligned} \tag{39}$$

$$\varepsilon_t = \frac{c}{(1+\theta)} + \sum_{i=0}^{\infty}\pi_i y_{t-i} \tag{40}$$

Where

$$\pi_i = \begin{cases} 1, & i = 0 \\ -(\theta + \phi_1), & i = 1 \\ \theta^2 + \phi_1\theta - \phi_2, & i = 2 \\ (-1)^i[\theta^i + \phi_1\theta^{i-1} - \phi_2\theta^{i-2} + \phi_3\theta^{i-3}], & i = 3, 4, \dots \end{cases} \quad (41)$$

Equation (40) holds if $\theta \neq -1$.

ARIMA (3, 1, 1)

$$\Delta y_t = c + \phi_1\Delta y_{t-1} + \phi_2\Delta y_{t-2} + \phi_3\Delta y_{t-3} + \varepsilon_t + \theta\varepsilon_{t-1} \quad (42)$$

$$(1 + \theta L)\varepsilon_t = y_t - (1 + \phi_1)y_{t-1} + (\phi_1 - \phi_2)y_{t-2} + (\phi_2 - \phi_3)y_{t-3} + \phi_3 y_{t-4} - c$$

$$\varepsilon_t = -(1 + \theta L)^{-1}c + (1 + \theta L)^{-1}[y_t - (1 + \phi_1)y_{t-1} + (\phi_1 - \phi_2)y_{t-2} + (\phi_2 - \phi_3)y_{t-3} + \phi_3 y_{t-4}]$$

$$\begin{aligned}\varepsilon_t = -\frac{c}{(1+\theta)} &+ y_t - (\theta + \phi_1 + 1)y_{t-1} + [\theta^2 + \theta(1 + \phi_1) + \phi_1 - \phi_2]y_{t-2} \\ &- [\theta^3 + \theta^2(1 + \phi_1) + \theta(\phi_1 - \phi_2) - \phi_2 + \phi_3]y_{t-3} \\ &+ \sum_{i=4}^{\infty}(-1)^i[\theta^i + \theta^{i-1}(1 + \phi_1) + \theta^{i-2}(\phi_1 - \phi_2) - \theta^{i-3}(\phi_2 - \phi_3) \\ &+ \theta^{i-4}\phi_3]y_{t-i} \end{aligned} \quad (43)$$

$$\varepsilon_t = -\frac{c}{(1+\theta)} + \sum_{i=0}^{\infty}\pi_i y_{t-i} \quad (44)$$

where

$$\pi_i = \begin{cases} 1, & i = 0 \\ -(\theta + \phi_1 + 1), & i = 1 \\ \theta^2 + \theta(1 + \phi_1) + \phi_1 - \phi_2, & i = 2 \\ -[\theta^3 + \theta^2(1 + \phi_1) + \theta(\phi_1 - \phi_2) - (\phi_2 - \phi_3)], & i = 3 \\ (-1)^i[\theta^i + \theta^{i-1}(1 + \phi_1) + \theta^{i-2}(\phi_1 - \phi_2) & \\ \qquad -\theta^{i-3}(\phi_2 - \phi_3) + \theta^{i-4}\phi_3], & i = 4, 5, 6, \dots \end{cases} \quad (45)$$

Equation (44) holds if $\theta \neq -1$.

ARIMA (3, 2, 1)

$$\Delta^2 y_t = c + \phi_1\Delta^2 y_{t-1} + \phi_2\Delta^2 y_{t-2} + \phi_3\Delta^2 y_{t-3} + \varepsilon_t + \theta\varepsilon_{t-1} \quad (46)$$

$$y_t - 2y_{t-1} + y_{t-2} = c + \phi_1(y_{t-1} - 2y_{t-2} + y_{t-3}) + \phi_2(y_{t-2} - 2y_{t-3} + y_{t-4}) + \phi_3(y_{t-3} - 2y_{t-4} + y_{t-5}) + \varepsilon_t + \theta\varepsilon_{t-1}$$

$$(1 + \theta L)\varepsilon_t = y_t - (2 + \phi_1)y_{t-1} + (1 + 2\phi_1 - \phi_2)y_{t-2} - (\phi_1 - 2\phi_2 + \phi_3)y_{t-3} - (\phi_2 - 2\phi_3)y_{t-4} - \phi_3 y_{t-5} - c$$

$$\begin{aligned}\varepsilon_t = -\frac{c}{(1+\theta)} &+ y_t - (\theta + \phi_1 + 2)y_{t-1} + [\theta^2 + \theta(2 + \phi_1) + (1 + 2\phi_1 - \phi_2)]y_{t-2} \\ &- [\theta^3 + \theta^2(2 + \phi_1) + \theta(1 + 2\phi_1 - \phi_2) + (\phi_1 - 2\phi_2 + \phi_3)]y_{t-3} \\ &+ [\theta^4 + \theta^3(2 + \phi_1) + \theta^2(1 + 2\phi_1 - \phi_2) + \theta(\phi_1 - 2\phi_2 + \phi_3) - (\phi_2 - 2\phi_3)]y_{t-4} \\ &+ \sum_{i=5}^{\infty}(-1)^i[\theta^i + \theta^{i-1}(\phi_1 + 2) + \theta^{i-2}(1 + 2\phi_1 - \phi_2) + \theta^{i-3}(\phi_1 - 2\phi_2 + \phi_3) - \theta^{i-4}(\phi_2 - 2\phi_3) \\ &+ \theta^{i-5}\phi_3]y_{t-i} \end{aligned} \quad (47)$$

$$\varepsilon_t = -\frac{c}{(1+\theta)} + \sum_{i=0}^{\infty} \pi_i y_{t-i} \quad (48)$$

where

$$\pi_i = \begin{cases} 1, & i=0 \\ -(\theta+\phi_1+2), & i=1 \\ \theta^2+\theta(2+\phi_1)+(1+2\phi_1-\phi_2), & i=2 \\ -[\theta^3+\theta^2(2+\phi_1)+\theta(1+2\phi_1-\phi_2)+(\phi_1-2\phi_2+\phi_3)], & i=3 \\ \theta^4+\theta^3(2+\phi_1)+\theta^2(1+2\phi_1-\phi_2) \\ \quad +\theta(\phi_1-2\phi_2+\phi_3)-(\phi_2-2\phi_3), & i=4 \\ (-1)^i[\theta^i+\theta^{i-1}(\phi_1+2)+\theta^{i-2}(1+2\phi_1-\phi_2)+\theta^{i-3}(\phi_1-2\phi_2+\phi_3) \\ \quad -\theta^{i-4}(\phi_2-2\phi_3)+\theta^{i-5}\phi_3], & i=4,5,6,\dots \end{cases} \quad (49)$$

Equation (48) holds if $\theta \neq -1$.

ARIMA (3, 3, 1)

$$\Delta^3 y_t = c + \phi_1\Delta^3 y_{t-1} + \phi_2\Delta^3 y_{t-2} + \phi_3\Delta^3 y_{t-3} + \varepsilon_t + \theta\varepsilon_{t-1} \quad (50)$$

$$\begin{aligned} y_t - 3y_{t-1} + 3y_{t-2} - y_{t-3} &= c + \phi_1(y_{t-1} - 3y_{t-2} + 3y_{t-3} - y_{t-4}) + \phi_2(y_{t-2} - 3y_{t-3} + 3y_{t-4} - y_{t-5}) \\ &\quad + \phi_3(y_{t-3} - 3y_{t-4} + 3y_{t-5} - y_{t-6}) + \varepsilon_t + \theta\varepsilon_{t-1} \end{aligned}$$

$$\begin{aligned} (1+\theta L)\varepsilon_t &= y_t - (3+\phi_1)y_{t-1} + (3+3\phi_1-\phi_2)y_{t-2} - (1+3\phi_1-3\phi_2+\phi_3)y_{t-3} + (\phi_1-3\phi_2+3\phi_3)y_{t-4} \\ &\quad + (\phi_2-3\phi_3)y_{t-5} + \phi_3 y_{t-6} - c \end{aligned}$$

$$\begin{aligned} \varepsilon_t = -\frac{c}{(1+\theta)} &+ y_t - (\theta+\phi_1+3)y_{t-1} + [\theta^2+\theta(3+\phi_1)+(3+3\phi_1-\phi_2)]y_{t-2} \\ &- [\theta^3+\theta^2(3+\phi_1)+\theta(3+3\phi_1-\phi_2)+(1+3\phi_1-3\phi_2+\phi_3)]y_{t-3} \\ &+ [\theta^4+\theta^3(3+\phi_1)+\theta^2(3+3\phi_1-\phi_2)+\theta(1+3\phi_1-3\phi_2+\phi_3)+(\phi_1-3\phi_2+3\phi_3)]y_{t-4} \\ &- [\theta^5+\theta^4(3+\phi_1)+\theta^3(3+3\phi_1-\phi_2)+\theta^2(1+3\phi_1-3\phi_2+\phi_3)+\theta(\phi_1-3\phi_2+3\phi_3) \\ &+ (\phi_2-3\phi_3)]y_{t-5} \\ &+ \sum_{i=6}^{\infty}(-1)^i[\theta^i+\theta^{i-1}(\phi_1+3)+\theta^{i-2}(3+3\phi_1-\phi_2)+\theta^{i-3}(1+3\phi_1-3\phi_2+\phi_3) \\ &+ \theta^{i-4}(\phi_1-3\phi_2+3\phi_3)-\theta^{i-5}(\phi_2-3\phi_3) \\ &+ \theta^{i-6}\phi_3]y_{t-i} \end{aligned} \quad (51)$$

$$\varepsilon_t = -\frac{c}{(1+\theta)} + \sum_{i=0}^{\infty} \pi_i y_{t-i} \quad (52)$$

where

$$\pi_i = \begin{cases} 1, & i=0 \\ -(\theta+\phi_1+3), & i=1 \\ \theta^2+\theta(3+\phi_1)+(3+3\phi_1-\phi_2), & i=2 \\ -[\theta^3+\theta^2(3+\phi_1)+\theta(3+3\phi_1-\phi_2)+(1+3\phi_1-3\phi_2)], & i=3 \\ \theta^4+\theta^3(3+\phi_1)+\theta^2(3+3\phi_1-\phi_2)+\theta(1+3\phi_1-3\phi_2+\phi_3) \\ \quad +(\phi_1-3\phi_2+3\phi_3), & i=4 \\ -[\theta^5+\theta^4(3+\phi_1)+\theta^3(3+3\phi_1-\phi_2)+\theta^2(1+3\phi_1-3\phi_2+\phi_3) \\ \quad +\theta(\phi_1-3\phi_2+3\phi_3)+(\phi_2-3\phi_3), & i=5 \\ (-1)^i[\theta^i+\theta^{i-1}(\phi_1+3)+\theta^{i-2}(3+3\phi_1-\phi_2) \\ \quad +\theta^{i-3}(1+3\phi_1-3\phi_2+\phi_3)+\theta^{i-4}(\phi_1-3\phi_2+3\phi_3) \\ \quad -\theta^{i-5}(\phi_2-3\phi_3)+\theta^{i-6}\phi_3], & i=6,7,8,\dots \end{cases} \quad (53)$$

Equation (52) holds if $\theta \neq -1$.

ARIMA (p, 0, 1)

$$y_t = c + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \phi_3 y_{t-3} + \dots + \phi_p y_{t-p} + \varepsilon_t + \theta\varepsilon_{t-1} \quad (54)$$

$$(1+\theta L)\varepsilon_t = -c + y_t - \phi_1 y_{t-1} - \phi_2 y_{t-2} - \phi_3 y_{t-3} - \phi_4 y_{t-4} - \dots - \phi_p y_{t-p}$$

$$\varepsilon_t = -(1+\theta L)^{-1}c + (1+\theta L)^{-1}(y_t - \phi_1 y_{t-1} - \phi_2 y_{t-2} - \phi_3 y_{t-3} - \phi_4 y_{t-4} - \dots - \phi_p y_{t-p})$$

$$\varepsilon_t = -\frac{c}{(1+\theta)} + y_t - (\theta+\phi_1)y_{t-1} + (\theta^2+\phi_1\theta-\phi_2)y_{t-2} - (\theta^3+\phi_1\theta^2-\phi_2\theta+\phi_3)y_{t-3} + (\theta^4+\phi_1\theta^3-\phi_2\theta^2+\phi_3\theta-\phi_4)y_{t-4} - \dots + \sum_{i=p}^{\infty}(-1)^i[\theta^i+\phi_1\theta^{i-1}-\phi_2\theta^{i-2}+\phi_3\theta^{i-3}-\phi_4\theta^{i-4} + \dots + (-1)^{p+1}\phi_p\theta^{i-p}]y_{t-i} \qquad (55)$$

$$\varepsilon_t = -\frac{c}{(1+\theta)} + \sum_{i=0}^{\infty}\pi_i y_{t-i} \qquad (56)$$

where

$$\pi_i = \begin{cases} 1, & i=0 \\ -(\theta+\phi_1), & i=1 \\ \theta^2+\phi_1\theta-\phi_2, & i=2 \\ -(\theta^3+\phi_1\theta^2-\phi_2\theta+\phi_3), & i=3 \\ \theta^4+\phi_1\theta^3-\phi_2\theta^2+\phi_3\theta-\phi_4, & i=4 \\ (-1)^i[\theta^i+\phi_1\theta^{i-1}-\phi_2\theta^{i-2}+\phi_3\theta^{i-3}-\phi_4\theta^{i-4}+ \\ \quad \dots+(-1)^{p+1}\phi_p\theta^{i-p}], & i=p,p+1,\dots \end{cases} \qquad (57)$$

Equation (56) holds if $\theta \neq -1$.

ARIMA (p, 1, 1)

$$\Delta y_t = c + \phi_1\Delta y_{t-1} + \phi_2\Delta y_{t-2} + \phi_3\Delta y_{t-3} + \dots + \phi_p\Delta y_{t-p} + \varepsilon_t + \theta\varepsilon_{t-1} \qquad (58)$$

$$(1+\theta L)\varepsilon_t = y_t - (1+\phi_1)y_{t-1} + (\phi_1-\phi_2)y_{t-2} + (\phi_2-\phi_3)y_{t-3} + (\phi_3-\phi_4)y_{t-4} + \dots + (\phi_{p-1}-\phi_p)y_{t-p} + \phi_p y_{t-p-1} - c$$

$$\varepsilon_t = -(1+\theta L)^{-1}c + (1+\theta L)^{-1}\big[y_t - (1+\phi_1)y_{t-1} + (\phi_1-\phi_2)y_{t-2} + (\phi_2-\phi_3)y_{t-3} + (\phi_3-\phi_4)y_{t-4} + \cdots + (\phi_{p-1}-\phi_p)y_{t-p} + \phi_p y_{t-p-1}\big]$$

$$\varepsilon_t = -\frac{c}{(1+\theta)} + y_t - (\theta+\phi_1+1)y_{t-1} + [\theta^2+\theta(1+\phi_1)+\phi_1-\phi_2]y_{t-2} - [\theta^3+\theta^2(1+\phi_1)+\theta(\phi_1-\phi_2)-\phi_2+\phi_3]y_{t-3} + [\theta^4+\theta^3(1+\phi_1)+\theta^2(\phi_1-\phi_2)-\theta(\phi_2-\phi_3)+\phi_3-\phi_4]y_{t-4} + \dots + \sum_{i=p}^{\infty}(-1)^i[\theta^i+\theta^{i-1}(1+\phi_1)+\theta^{i-2}(\phi_1-\phi_2)-\theta^{i-3}(\phi_2-\phi_3)+\theta^{i-4}(\phi_3-\phi_4)+\cdots + (-1)^{p+1}\theta^{i-p}\phi_p]y_{t-i} \qquad (59)$$

$$\varepsilon_t = -\frac{c}{(1+\theta)} + \sum_{i=0}^{\infty}\pi_i y_{t-i} \qquad (60)$$

where

$$\pi_i = \begin{cases} 1, & i=0 \\ -(\theta+\phi_1+1), & i=1 \\ \theta^2+\theta(1+\phi_1)+\phi_1-\phi_2, & i=2 \\ -[\theta^3+\theta^2(1+\phi_1)+\theta(\phi_1-\phi_2)-\phi_2+\phi_3], & i=3 \\ (-1)^i[\theta^i+\theta^{i-1}(1+\phi_1)+\theta^{i-2}(\phi_1-\phi_2)-\theta^{i-3}(\phi_2-\phi_3) \\ +\theta^{i-4}(\phi_3-\phi_4)-\theta^{i-5}(\phi_4-\phi_5)+\cdots+(-1)^{p+1}\theta^{i-p-1}\phi_p], & i=p+1,\dots \end{cases} \qquad (61)$$

Equation (60) holds if $\theta \neq -1$.

ARIMA (p, 2, 1)

$$\Delta^2 y_t = c + \phi_1\Delta^2 y_{t-1} + \phi_2\Delta^2 y_{t-2} + \phi_3\Delta^2 y_{t-3} + \dots + \phi_p\Delta^2 y_{t-p} + \varepsilon_t + \theta\varepsilon_{t-1} \qquad (62)$$

$$y_t - 2y_{t-1} + y_{t-2} = c + \phi_1(y_{t-1}-2y_{t-2}+y_{t-3}) + \phi_2(y_{t-2}-2y_{t-3}+y_{t-4}) + \phi_3(y_{t-3}-2y_{t-4}+y_{t-5}) + \cdots + \phi_p(y_{t-p} - 2y_{t-p-1} + y_{t-p-2}) + \varepsilon_t + \theta\varepsilon_{t-1}$$

$$(1+\theta L)\varepsilon_t = y_t - (2+\phi_1)y_{t-1} + (1+2\phi_1-\phi_2)y_{t-2} - (\phi_1-2\phi_2+\phi_3)y_{t-3} - (\phi_2-2\phi_3+\phi_4)y_{t-4} - \dots - (\phi_{p-1}-2\phi_p)y_{t-p-1} - \phi_p y_{t-p-2} - c$$

$$\varepsilon_t = -\frac{c}{(1+\theta)} + y_t - (\theta+\phi_1+2)y_{t-1} + [\theta^2+\theta(2+\phi_1)+(1+2\phi_1-\phi_2)]y_{t-2}$$
$$- [\theta^3+\theta^2(2+\phi_1)+\theta(1+2\phi_1-\phi_2)+(\phi_1-2\phi_2+\phi_3)]y_{t-3}$$
$$+ [\theta^4+\theta^3(2+\phi_1)+\theta^2(1+2\phi_1-\phi_2)+\theta(\phi_1-2\phi_2+\phi_3)-(\phi_2-2\phi_3+\phi_4)]y_{t-4} - \ \dots$$
$$+ (-1)^{p+1}\big[\theta^{p+1}+\theta^p(\phi_1+2)+\theta^{p-1}(1+2\phi_1-\phi_2)+\theta^{p-2}(\phi_1-2\phi_2+\phi_3)$$
$$- \theta^{p-3}(\phi_2-2\phi_3+\phi_4) + \ \dots + \theta\big(\phi_{p-2}-2\phi_{p-1}+\phi_p\big) + (-1)^{p+1}\big(\phi_{p-1}-2\phi_p\big)\big]y_{t-p-1}$$
$$+ \sum_{i=p+2}^{\infty} (-1)^i[\theta^i+\theta^{i-1}(\phi_1+2)+\theta^{i-2}(1+2\phi_1-\phi_2)+\theta^{i-3}(\phi_1-2\phi_2+\phi_3)$$
$$- \theta^{i-4}(\phi_2-2\phi_3+\phi_4) + \ \dots + (-1)^p\theta^{i-p-1}\big(\phi_{p-1}-2\phi_p\big) + (-1)^{p+1}\theta^{i-p-2}\phi_p]y_{t-i} \qquad (63)$$

$$\varepsilon_t = -\frac{c}{(1+\theta)} + \sum_{i=0}^{\infty}\pi_i y_{t-i} \qquad (64)$$

where

$$\pi_i = \begin{cases} 1, & i=0 \\ -(\theta+\phi_1+2), & i=1 \\ \theta^2+\theta(2+\phi_1)+(1+2\phi_1-\phi_2), & i=2 \\ -[\theta^3+\theta^2(2+\phi_1)+\theta(1+2\phi_1-\phi_2)+(\phi_1-2\phi_2+\phi_3)], & i=3 \\ \theta^4+\theta^3(2+\phi_1)+\theta^2(1+2\phi_1-\phi_2) & \\ \qquad +\theta(\phi_1-2\phi_2+\phi_3)-(\phi_2-2\phi_3+\phi_4), & i=4 \\ -[\theta^5+\theta^4(2+\phi_1)+\theta^3(1+2\phi_1-\phi_2)+\theta^2(\phi_1-2\phi_2+\phi_3) & \\ \qquad -\theta(\phi_2-2\phi_3+\phi_4)+(\phi_3-2\phi_4+\phi_5)], & i=5 \\ (-1)^{p+1}[\theta^{p+1}+\theta^p(\phi_1+2)+\theta^{p-1}(1+2\phi_1-\phi_2) & \\ \qquad +\theta^{p-2}(\phi_1-2\phi_2+\phi_3)-\theta^{p-3}(\phi_2-2\phi_3+\phi_4)+\ \dots & \\ \qquad +\theta\big(\phi_{p-2}-2\phi_{p-1}+\phi_p\big)+(-1)^{p+1}\big(\phi_{p-1}-2\phi_p\big)], & i=p+1 \\ (-1)^i[\theta^i+\theta^{i-1}(\phi_1+2)+\theta^{i-2}(1+2\phi_1-\phi_2) & \\ \qquad -\theta^{i-4}(\phi_2-2\phi_3+\phi_4)+\cdots+(-1)^p\theta^{i-p-1}\big(\phi_{p-1}-2\phi_p\big) & \\ \qquad +(-1)^{p+1}\theta^{i-p-2}\phi_p], & i=p+2,p+3,\dots \end{cases} \qquad (65)$$

Equation (64) holds if $\theta \neq -1$.

ARIMA (p, 3, 1)

$$\Delta^3 y_t = c + \phi_1\Delta^3 y_{t-1} + \phi_2\Delta^3 y_{t-2} + \phi_3\Delta^3 y_{t-3} + \ \dots + \phi_p\Delta^3 y_{t-p} + \varepsilon_t + \theta\varepsilon_{t-1} \qquad (66)$$

$$y_t - 3y_{t-1} + 3y_{t-2} - y_{t-3}$$
$$= c + \phi_1(y_{t-1}-3y_{t-2}+3y_{t-3}-y_{t-4}) + \phi_2(y_{t-2}-3y_{t-3}+3y_{t-4}-y_{t-5})$$
$$+ \phi_3(y_{t-3}-3y_{t-4}+3y_{t-5}-y_{t-6}) + \ \dots + \phi_p\big(y_{t-p}-3y_{t-p-1}+3y_{t-p-2}-y_{t-p-3}\big) + \varepsilon_t + \theta\varepsilon_{t-1}$$

$$(1+\theta L)\varepsilon_t = y_t - (3+\phi_1)y_{t-1} + (3+3\phi_1-\phi_2)y_{t-2} - (1+3\phi_1-3\phi_2+\phi_3)y_{t-3} + (\phi_1-3\phi_2+3\phi_3-\phi_4)y_{t-4}$$
$$- (\phi_2-3\phi_3+3\phi_4-\phi_5)y_{t-5} + \ \dots + \big(\phi_{p-3}-3\phi_{p-2}+3\phi_{p-1}-\phi_p\big)y_{t-p}$$
$$+ \big(\phi_{p-2}-3\phi_{p-1}+3\phi_p\big)y_{t-p-1} + (\phi_{p-1}-3\phi_p)y_{t-p-2} + \phi_p y_{t-p-3} - c$$

$$\varepsilon_t = -(1+\theta L)^{-1}c$$
$$+ (1+\theta L)^{-1}\big[y_t - (3+\phi_1)y_{t-1} + (3+3\phi_1-\phi_2)y_{t-2} - (1+3\phi_1-3\phi_2+s\phi_3)y_{t-3}$$
$$+ (\phi_1-3\phi_2+3\phi_3-\phi_4)y_{t-4} - (\phi_2-3\phi_3+3\phi_4-\phi_5)y_{t-5} + \ \dots$$
$$+ \big(\phi_{p-3}-3\phi_{p-2}+3\phi_{p-1}-\phi_p\big)y_{t-p} + \big(\phi_{p-2}-3\phi_{p-1}+3\phi_p\big)y_{t-p-1} + (\phi_{p-1}-3\phi_p)y_{t-p-2}$$
$$+ \phi_p y_{t-p-3}\big]$$

$$\varepsilon_t = -\frac{c}{(1+\theta)} + y_t - (\theta + \phi_1 + 3)y_{t-1} + [\theta^2 + \theta(3+\phi_1) + (3 + 3\phi_1 - \phi_2)]y_{t-2}$$
$$- [\theta^3 + \theta^2(3+\phi_1) + \theta(3 + 3\phi_1 - \phi_2) + (1 + 3\phi_1 - 3\phi_2 + \phi_3)]y_{t-3}$$
$$+ [\theta^4 + \theta^3(3+\phi_1) + \theta^2(3 + 3\phi_1 - \phi_2) + \theta(1 + 3\phi_1 - 3\phi_2 + \phi_3) + (\phi_1 - 3\phi_2 + 3\phi_3 - \phi_4)]y_{t-4}$$
$$+ \ \dots$$
$$+ (-1)^{p+2}\big[\theta^{p+2} + \theta^{p+1}(3+\phi_1) + \theta^p(3 + 3\phi_1 - \phi_2) + \theta^{p-1}(1 + 3\phi_1 - 3\phi_2 + \phi_3) + \ \dots$$
$$+ (-1)^p\theta^2\big(\phi_{p-3} - 3\phi_{p-2} + 3\phi_{p-1} - \phi_p\big) + (-1)^{p+1}\theta\big(\phi_{p-2} - 3\phi_{p-1} + 3\phi_p\big)$$
$$+ (-1)^{p+2}\big(\phi_{p-1} - 3\phi_p\big)\big]y_{t-p-2}$$
$$+ \sum_{i=p+3}^{\infty} (-1)^i[\theta^i + \theta^{i-1}(\phi_1 + 3) + \theta^{i-2}(3 + 3\phi_1 - \phi_2) + \theta^{i-3}(1 + 3\phi_1 - 3\phi_2 + \phi_3)$$
$$+ \theta^{i-4}(\phi_1 - 3\phi_2 + 3\phi_3 - \phi_4) + \ \dots + (-1)^{p+1}\theta^{i-p-1}\big(\phi_{p-2} - 3\phi_{p-1} + 3\phi_p\big)$$
$$+ (-1)^{p+2}\theta^{i-p-2}\big(\phi_{p-1} - 3\phi_p\big)$$
$$+ (-1)^{p+3}\theta^{i-p-3}\phi_p]y_{t-i} \qquad (67)$$

$$\varepsilon_t = -\frac{c}{(1+\theta)} + \sum_{i=0}^{\infty}\pi_i y_{t-i} \qquad (68)$$

where

$$\pi_i = \begin{cases} 1, & i = 0 \\ -(\theta + \phi_1 + 3), & i = 1 \\ \theta^2 + \theta(3+\phi_1) + (3 + 3\phi_1 - \phi_2), & i = 2 \\ -[\theta^3 + \theta^2(3+\phi_1) + \theta(3 + 3\phi_1 - \phi_2) + (1 + 3\phi_1 - 3\phi_2 + \phi_3)], & i = 3 \\ \theta^4 + \theta^3(3+\phi_1) + \theta^2(3 + 3\phi_1 - \phi_2) & \\ \quad +\theta(1 + 3\phi_1 - 3\phi_2 + \phi_3) + (\phi_1 - 3\phi_2 + 3\phi_3 - \phi_4), & i = 4 \\ (-1)^{p+2}[\theta^{p+2} + \theta^{p+1}(3+\phi_1) + \theta^p(3 + 3\phi_1 - \phi_2) + & \\ \quad \dots + \theta\big(\phi_{p-2} - 3\phi_{p-1} + 3\phi_p\big) + \big(\phi_{p-1} - 3\phi_p\big)], & i = p+2 \\ (-1)^i[\theta^i + \theta^{i-1}(\phi_1 + 3) + \theta^{i-2}(3 + 3\phi_1 - \phi_2) & \\ \quad +\theta^{i-3}(1 + 3\phi_1 - 3\phi_2 + \phi_3) + \ \dots + \theta^{i-p-1}\big(\phi_{p-2} - 3\phi_{p-1} + 3\phi_p\big) & \\ \quad +\theta^{i-p-2}\big(\phi_{p-1} - 3\phi_p\big) + \theta^{i-p-3}\phi_p], & i = p+3, p+4, \dots \end{cases} \qquad (69)$$

Equation (68) holds if $\theta \neq -1$.
The expression for ARIMA (*p*, *d*, *1*) for various values of $p \geq 1$ holds if $\theta \neq -1$ for the case where constant term c exists but holds for all values of θ if the is no constant term.

Behavioural pattern of invertibility parameters of ARIMA (*p*, *d*, *1*) model

Here we consider the behaviour of π_i for some *i* for some orders of ϕ_i where $i = 1, 2, 3, \dots, p$.

(a) Given that $\phi_1 > \phi_2 > \phi_3 > \ \dots \ > \phi_p > 0$, we examine the behavioural pattern of π_i for both positive and negative values of θ and obtain the following result for $d = 0,\ 1, 2$ and 3. Given that $\phi_1 > \phi_2 > \phi_3 > \cdots > \phi_p > 0$ for various values of *p*, the invertibility parameter π_i of ARIMA (*p*, *d*, 1) are sinusoidal and the absolute value of the invertibility parameter, $|\pi_i|$ increases as *d* increases for positive values of θ. That is $|\pi_i|_{d=3} > |\pi_i|_{d=2} > |\pi_i|_{d=1} > |\pi_i|_{d=0}$ which implies that the lower the integer value of *d*, the faster $|\pi_i|$ converges to zero as shown in Figure 1 below. For negative integer values of θ, it is seen from Figure 2 that the invertibility parameter π_i of ARIMA (*p*, *d*, 1) model tends to oscillate as the value of *d* increases with $|\pi_i|_{d=k} < |\pi_i|_{d=k+2}$ for all even integer value $k > 0$ provided $i \geq 3$ and $|\pi_i|_{d=k} > |\pi_i|_{d=k+2}$ for all odd integer value $k > 0$.

The invertibility parameter π_i of ARIMA (*p*, *0*, 1) model is sinusoidal for $p = 1, 2, 3$ and 4 with $|\pi_i|_{p=4} > |\pi_i|_{p=3} > |\pi_i|_{p=2} > |\pi_i|_{p=1}$ for all positive values of θ as seen in Figure 3. For all negative values of θ, $|\pi_i|_{p=k} < |\pi_i|_{p=k+2}$ for odd integer value $k > 0$ and $|\pi_i|_{p=k} < |\pi_i|_{p=k+2}$ for even integer value $k > 0$ as seen in Figure 4. Similarly, invertibility parameter π_i of ARIMA (*p*, *1*, 1) model is sinusoidal for $p = 1, 2, 3$ and 4 with $|\pi_i|_{p=4} > |\pi_i|_{p=3} > |\pi_i|_{p=2} > |\pi_i|_{p=1}$ for all positive values of θ as seen in Figure 3. For all negative values of θ, $|\pi_i|_{p=k} < |\pi_i|_{p=k+2}$ for odd integer value $k > 0$ and $|\pi_i|_{p=k} < |\pi_i|_{p=k+2}$ for even integer value $k > 0$ as seen in Figure 6.

(b) Given that $\phi_1 = \phi_2 = \phi_3 = \ ... = \phi_p > 0$, we examine the behavioural pattern of π_i for positive of θ and obtain the following result for $d = 0, 1, 2$ and 3, the invertibility parameter π_i of ARIMA (*p*, *d*, 1) are sinusoidal and the absolute value of the invertibility parameter, $|\pi_i|$ increases as d increases for positive values of θ. That is $|\pi_i|_{d=3} > |\pi_i|_{d=2} > |\pi_i|_{d=1} > |\pi_i|_{d=0}$ which implies that the lower the integer value of *d*, the faster $|\pi_i|$ converges to zero as shown in Figure 7 below.

(c) We examine the behavioural pattern of π_i for positive of θ and obtain the following result for $d = 0, 1, 2$ and 3. Given that $0 < \phi_1 < \phi_2 < \phi_3 < \ ... < \phi_p$ for various values of *p*, the invertibility parameter π_i of ARIMA (*4*, *d*, 1) are sinusoidal and the absolute value of the invertibility parameter, $|\pi_i|$ increases as *d* increases for positive values of θ. That is $|\pi_i|_{d=3} > |\pi_i|_{d=2} > |\pi_i|_{d=1} > |\pi_i|_{d=0}$ which implies that the lower the integer value of *d*, the faster $|\pi_i|$ converges to zero as shown in Figure 8 below.

(d) We examine the behavioural pattern of π_i for different ϕ_p's arrangement, it is seen in Figure 9 that $|\pi_i|$ is the greatest for decreasing order of ϕ_p (i.e. $\phi_1 > \phi_2 > \phi_3 > \cdots > \phi_p > 0$), follow by for equal order of ϕ_p ($\phi_1 = \phi_2 = \phi_3 = \ ... = \phi_p > 0$) and least for increasing order of ϕ_p ($0 < \phi_1 < \phi_2 < \phi_3 < \ ... < \phi_p$).

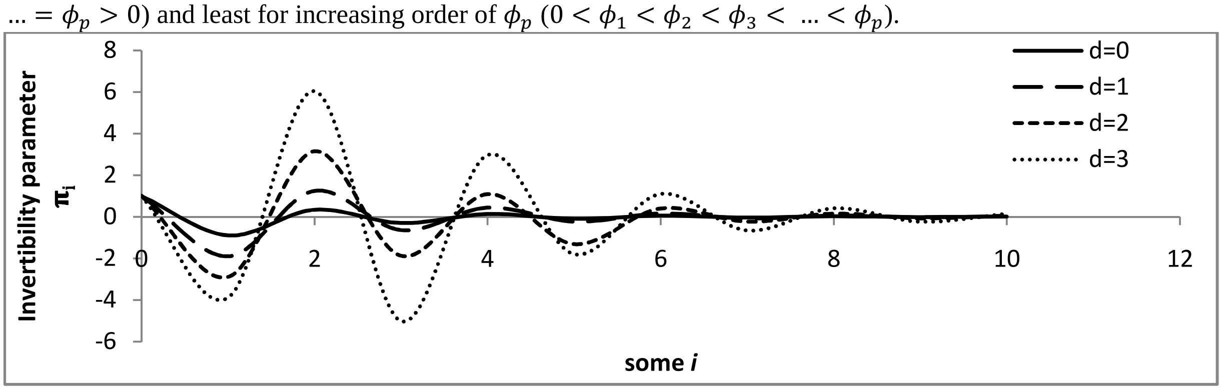

Figure 1: Graph showing behaviour of invertibility parameter π_i of ARIMA (*p*, *d*, *1*) for some *i* when $d = 0, 1, 2$ and 3, given that $\theta = 0.6$, $\phi_1 = 0.3$, $\phi_2 = 0.2$, $\phi_3 = 0.1$, $\phi_4 = 0.05$, $\phi_5 = 0.025$, $\phi_6 = 0.0125$, $\phi_7 = 0.00625$, $\phi_8 = 0.003125$, $\phi_9 = 0.0015625$ and $\phi_{10} = 0.00078125$.

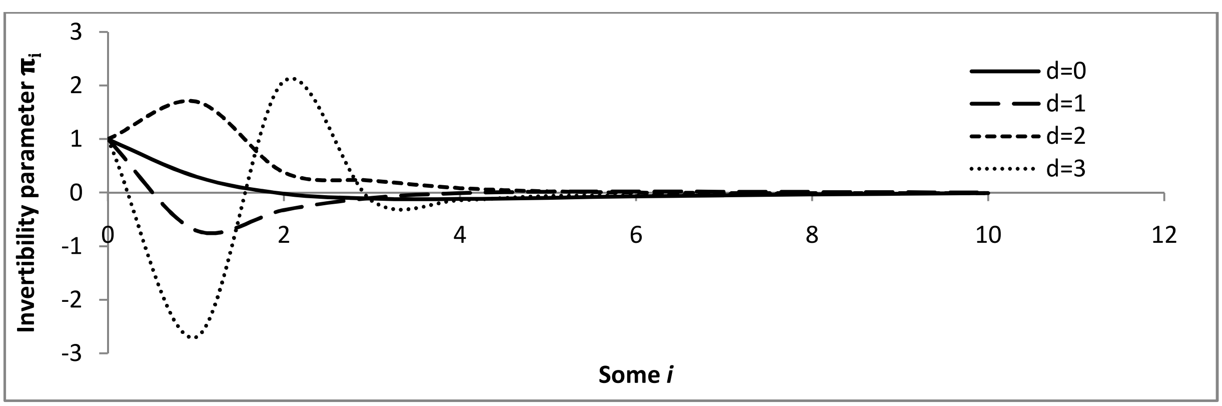

Figure 2: Graph showing behaviour of invertibility parameter π_i of ARIMA (*p*, *d*, *1*) for some *i* when $d = 0, 1, 2$ and 3, given that $\theta = -0.6$, $\phi_1 = 0.3$, $\phi_2 = 0.2$, $\phi_3 = 0.1$, $\phi_4 = 0.05$, $\phi_5 = 0.025$, $\phi_6 = 0.0125$, $\phi_7 = 0.00625$, $\phi_8 = 0.003125$, $\phi_9 = 0.0015625$ and $\phi_{10} = 0.00078125$.

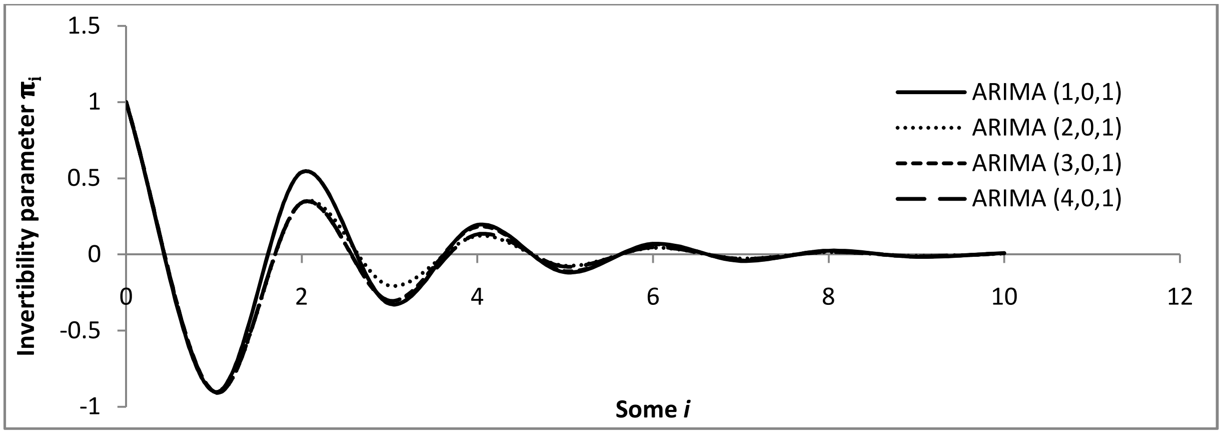

Figure 3: Graph showing behaviour of invertibility parameter π_i of ARIMA (*p*, *0*, *1*) for some *i* when $p = 1, 2, 3$ and 4, given that $\theta = 0.6$, $\phi_1 = 0.3$, $\phi_2 = 0.2$, $\phi_3 = 0.1$ and $\phi_4 = 0.05$.

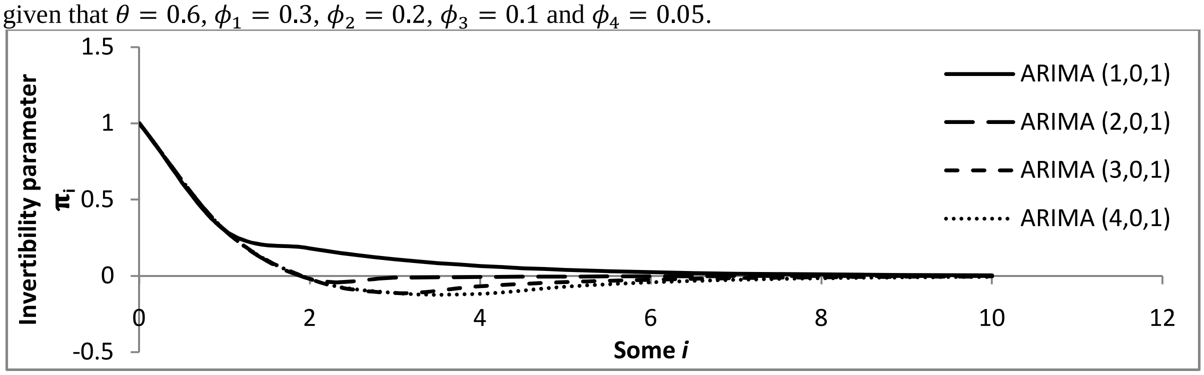

Figure 4: Graph showing behaviour of invertibility parameter π_i of ARIMA (*p*, *0*, *1*) for some *i* when $p = 1, 2, 3$ and 4, given that $\theta = -0.6$, $\phi_1 = 0.3$, $\phi_2 = 0.2$, $\phi_3 = 0.1$ and $\phi_4 = 0.05$.

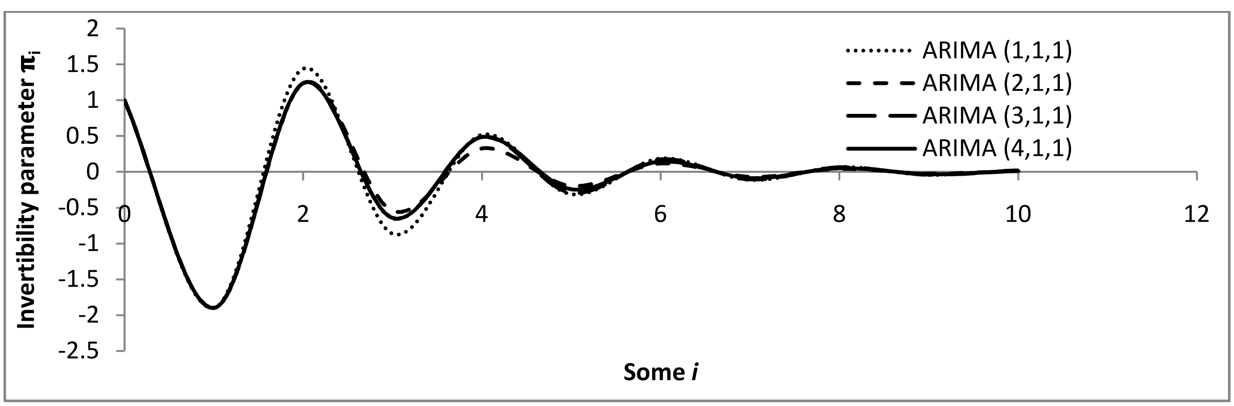

Figure 5: Graph showing behaviour of invertibility parameter π_i of ARIMA (*p*, *1*, *1*) for some *i* when $p = 1, 2, 3$ and 4, given that $\theta = 0.6$, $\phi_1 = 0.3$, $\phi_2 = 0.2$, $\phi_3 = 0.1$ and $\phi_4 = 0.05$.

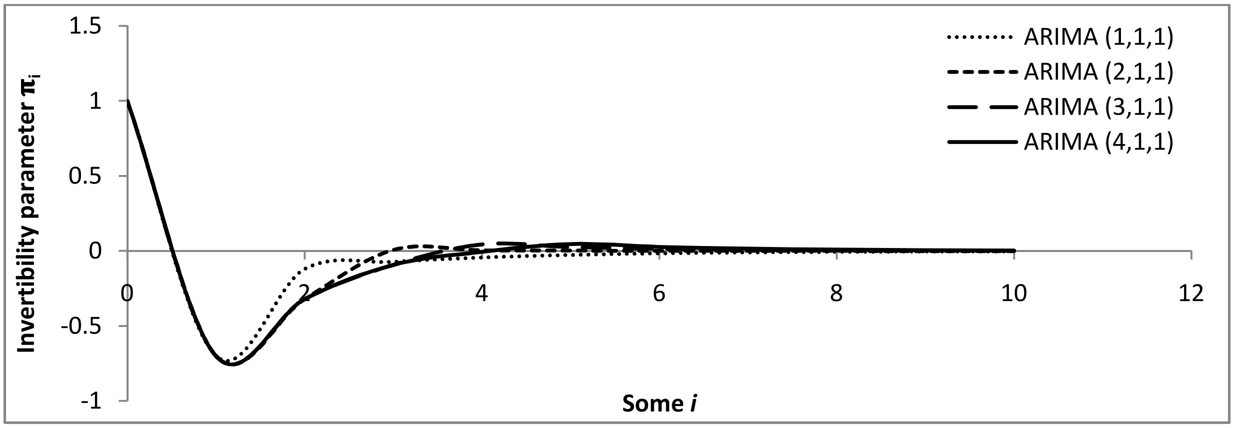

Figure 6: Graph showing behaviour of invertibility parameter π_i of ARIMA (*p*, *1*, *1*) for some *i* when $p = 1, 2, 3$ and 4, given that $\theta = -0.6$, $\phi_1 = 0.3$, $\phi_2 = 0.2$, $\phi_3 = 0.1$ and $\phi_4 = 0.05$.

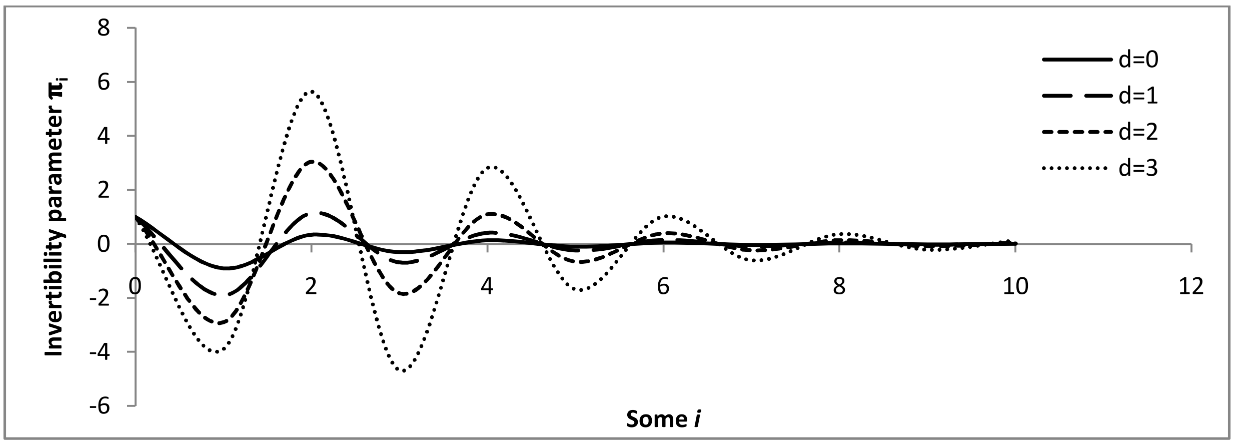

Figure 7: Graph showing behaviour of invertibility parameter π_i of ARIMA (*p*, *d*, *1*) for some *i* when $d = 0, 1, 2$ and 3, given that $\theta = 0.6$, $\phi_1 = \phi_2 = \ldots = \phi_{10} = 0.3$.

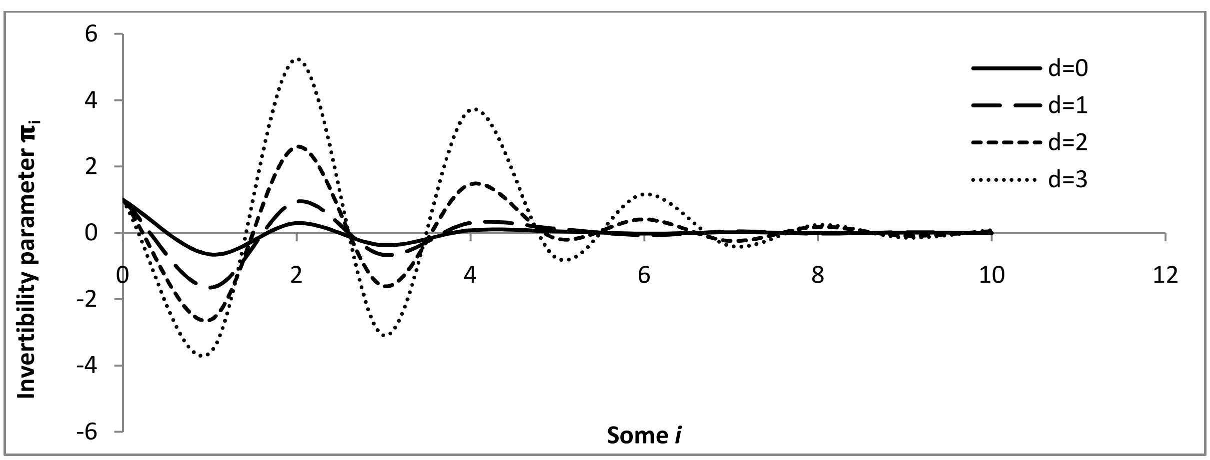

Figure 8: Graph showing behaviour of invertibility parameter π_i of ARIMA (*4*, *d*, *1*) for some *i* when $d = 0, 1, 2$ and 3, given that $\theta = 0.6$, $\phi_1 = 0.05$, $\phi_2 = 0.1$, $\phi_3 = 0.2$ and $\phi_4 = 0.3$.

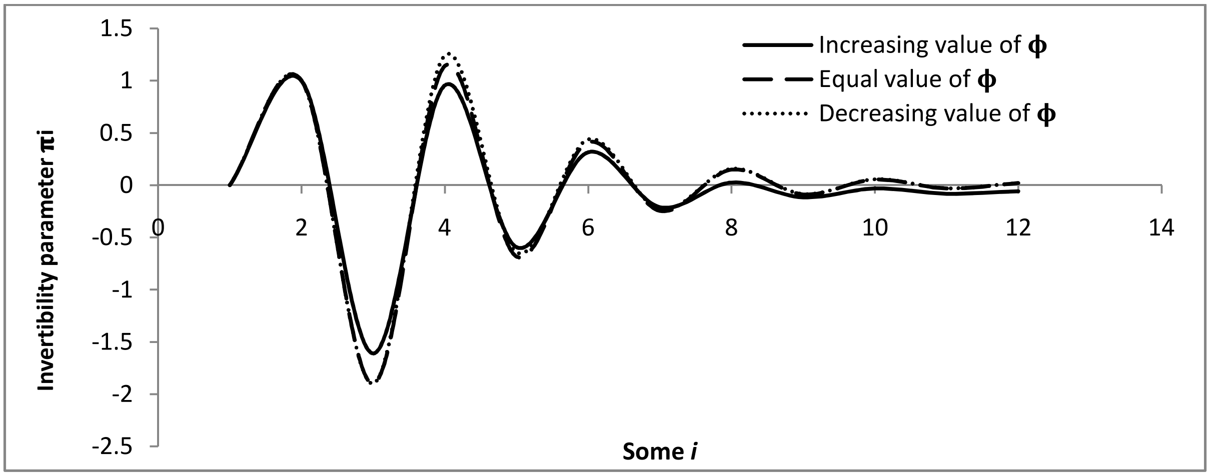

Figure 9: Graph showing behaviour of invertibility parameter π_i of ARIMA (*p, d, 1*) for some *i* under different ϕ_p's arrangement.

Conclusion

In this study, ARIMA model of various orders was presented in inverted form, behavioural pattern of invertibility parameter was investigated. It was deduced that behaviour of invertibility parameter π_i depend on positive and negative values of moving average parameter θ. For various sequential arrangement of $\phi_1, \phi_2, \phi_3, \ldots, \phi_p > 0$ for all integer values of *p*, it was deduced that the invertibility parameter π_i of ARIMA (*p*, *d*, 1) for various integer values of *d* are sinusoidal, the absolute value of the invertibility parameter, $|\pi_i|$ increases as *d* increases for positive values of θ and the lower the integer value of *d*, the faster $|\pi_i|$ converges to zero as shown in Figures 1, 7, 8 above. For negative integer values of θ, it was shown in Figure 2 that the invertibility parameter π_i of ARIMA (*p*, *d*, 1) model oscillates as the value of *d* increases with$|\pi_i|_{d=k} < |\pi_i|_{d=k+2}$ for all even integer value $k > 0$ provided $i \geq 3$ and $|\pi_i|_{d=k} > |\pi_i|_{d=k+2}$ for all odd integer value $k > 0$. Similarly, the invertibility parameter π_i of ARIMA (*p*, *d*, 1) model was found to be sinusoidal for $p = 1, 2, 3$ and 4 and $d = 1$ and 2 with $|\pi_i|_{p=4} > |\pi_i|_{p=3} > |\pi_i|_{p=2} > |\pi_i|_{p=1}$ for all positive values of θ as seen in Figures 3 and 5 above. For all negative values of θ, $|\pi_i|_{p=k} < |\pi_i|_{p=k+2}$ for odd integer value $k > 0$ and $|\pi_i|_{p=k} < |\pi_i|_{p=k+2}$ for even integer value $k > 0$ as seen in Figures 4 and 6.

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