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Behavioural Pattern of Causality Parameter of Autoregressive Moving Average Model

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Abstract

In this paper, a causal form of Autoregressive Moving Average process, ARMA (p, q) of various orders and behaviour of the causality parameter of ARMA model is investigated. It is deduced that the behaviour of causality parameter ψ_i depends on positive and negative values of autoregressive parameter ϕ and moving average parameter θ. The causality parameter is skewed to the right for positive values of ϕ and sinusoidal for negative values of ϕ while invertibility parameter is sinusoidal for positive values of θ.

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Keyword: invertibility, causality, stationarity, autoregressive

1.0 Introduction

When fitting ARMA model, we must check whether stationarity, causality and invertibility conditions are satisfied. If the pure AR form of time series is of finite order, the pure MA form will be of infinite order; if the pure MA form is of finite order the pure AR is of infinite order. If both the AR and MA parts of an ARMA are of finite order, both the pure AR and the pure MA are of infinite order. Therefore while a model has various representations it makes sense to look for the simplest representations to estimate.

Over the last years there has been growing interest in graphical models and in particular in those based on directed acyclic graphs as a general framework to describe and infer causal relations [1]. This new graphical approach is related to other approaches to formalize the concept of causality such as Neyman and Rubin's potential-response model [2] and path analysis or structural equation models [3]. The latter concept has been applied in particular by economists to describe the equilibrium distributions of systems which typically evolve over time.

Anderson [4] deduced conditions for the general Moving Average process, of order q, to be invertible or borderline non-invertible. He termed the conditions as acceptability conditions. It turned out that they depended on the magnitude of the final moving average parameter, θ_q. If $|\theta_q| > 1$, the process is not acceptable. Should $|\theta_q| = 1$, the conditions, for any particular q, follow simply if use is made of the remainder theorem. When $|\theta_q| < 1$, an appeal was made to ROUCH* E'S theorem, to establish the conditions. Analogous stationarity results immediately follow for autoregressive processes.

Mikosch et al [5] considered ARMA process of the form $\phi(L)X_t = \theta(L)Z_t$, where the innovations Z_t belong to the domain of attraction of a stable law, so that neither the X_t have a finite variance. They estimated the coefficients of ϕ and θ using Whittle estimator based on sample periodogram of X sequence. They showed that their estimators were consistent, obtained their asymptotic distributions and showed that they converged to the true values faster than in the usual $\mathfrak{L}^2$ case.

In this paper, ARIMA model of various orders are presented in causal and inverted form, behaviour of causality and invertibility parameters are investigated and the parameters of ARIMA were evaluated for various values of p and c at $q = 0$ using ordinary least squares method and Crammer's rule.

2. Autoregressive Moving Average Process

The need for estimating the parameters of an ARMA (p, q) process arises in many applications both in signal processing and in automatic control. One subset of ARMA models are the so-called autoregressive, or AR models while the other is moving average or MA models. The notation ARMA (p, q) refers to the model with p autoregressive terms and q moving average terms. This model contains the AR (p) and MA (q) models,

$$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \ ... + \phi_p y_{t-p} + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2} + \ ... + \theta_q \varepsilon_{t-q} \quad (1)$$

The terms $\phi_1 y_{t-1}$ through $\phi_p y_{t-p}$ are the autoregressive portion of the filter. The terms ε_t through $\theta_q \varepsilon_{t-q}$ are a moving average of the white noise input process. Autoregressive integrated moving average model (ARIMA) is a generalisation of ARMA model. It consists of Autoregressive model (AR), Integrated part (I; which is differencing term) and moving average

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model (MA). It is employed when time series is non stationary. p is the order of AR, d is the number of times a series is differenced to assume stationarity and q is the order of MA.

Specification of ARMA models in terms of lag operator

When the models are specified in terms of the lag operator L, the AR (p) model is given by $\varepsilon_t = \left(1 - \sum_{i=1}^{p} \phi_i L^i\right) y_t = \phi(L) y_t$, where $\phi(L) = 1 - \sum_{i=1}^{p} \phi_i L^i$ and MA (q) model is given by $y_t = \left(1 + \sum_{i=1}^{p} \theta_i L^i\right)\varepsilon_t = \theta(L)\varepsilon_t$, where $\theta(L) = 1 + \sum_{i=1}^{p} \theta_i L^i$. ARMA (p, q) is given as

$$\left(1 - \sum_{i=1}^{p} \phi_i L^i\right) y_t = \left(1 + \sum_{i=1}^{p} \theta_i L^i\right)\varepsilon_t \tag{2}$$

or more concisely,$\phi(L) y_t = \theta(L)\varepsilon_t$ which implies $y_t = \psi(L)\varepsilon_t$, where

$$\psi(L) = \frac{\theta(L)}{\phi(L)} = \frac{1+\theta_1 L + \theta_2 L^2 + \ldots + \theta_p L^p}{1-\phi_1 L - \phi_2 L^2 - \ldots - \phi_p L^p} \tag{3}$$

The ARMA process is stationary if

$$\sum_{j=1}^{\infty} \left|\psi_j\right| < \infty.$$

This happens if the series $\psi(Z)$ converges for every Z with $|Z| \le 1$. Since $\psi(Z)$ is a rational function, the series converges for every Z with $|Z| \le 1$ if the complex zeros of $\phi(Z)$ lie outside the unit circle. If we have a stationary process, then since $y_t = \psi(L)\varepsilon_t$, and the expected values of ε_t are all 0, the expected value of y_t is also 0.

An ARMA process y_t is invertible (strictly, an invertible function of ε_t) if there is a

$$\pi(L) = \pi_0 + \pi_1 L + \pi_2 L^2 + \_\_\_$$

with

$$\sum_{j=0}^{\infty} \left|\pi_j\right| < \infty$$

and $\varepsilon_t = \pi(L) y_t$.

An ARMA process y_t is causal if [7] there is a

$$\psi(L) = \psi_0 + \psi_1 L + \psi_2 L^2 + \_\_\_$$

with $\sum_{j=0}^{\infty} \left|\psi_j\right| < \infty$ and $y_t = \psi(L)\varepsilon_t$.

3. Causality of ARMA processes

An ARMA (*p*, *q*) process is causal if the absolute value of the parameters of ARMA (*p*, *q*) model satisfy $|\phi_i| < 1$ for $i = 1, \ldots, p$. If an ARMA process is causal, it is stationary. If ϕ and θ have no common factors, a (unique) *stationary* solution to $\phi(L) y_t = \theta(L)\varepsilon_t$ exists if and only if $|Z| = 1$ implies $\phi(Z) = 1 - \sum_{i=1}^{p} \phi_i Z^i \ne 0$. The ARMA (p, q) process is *causal* if and only if $|Z| \le 1$ implies $\phi(Z) = 1 - \sum_{i=1}^{p} \phi_i Z^i \ne 0$. It is *invertible* if and only if $|Z| \le 1$ implies $\theta(Z) = 1 + \sum_{i=1}^{q} \theta_i Z^i \ne 0$.

Presentation of some ARMA processes in causal form

Here we present some ARMA process of various orders in causal form of the process in order to provide a useful way of generating a random sequence. That is, we show a linear process y_t as a linear combination of white noise variates ε_t.

ARMA (1, 1)

$$y_t = c + \phi y_{t-1} + \varepsilon_t + \theta\varepsilon_{t-1} \tag{4}$$

$$y_t = \frac{c}{(1-\phi)} + \varepsilon_t + (\phi+\theta)\varepsilon_{t-1} + (\phi^2 + \theta\phi)\varepsilon_{t-2} + (\phi^3 + \theta\phi^2)\varepsilon_{t-3} + \ldots$$

$$y_t = \frac{c}{(1-\phi)} + \varepsilon_t + \sum_{i=1}^{\infty}(\phi^i + \theta\phi^{i-1})\varepsilon_{t-i}, \quad i = 1, 2, \ldots \tag{5}$$

$$y_t = \frac{c}{(1-\phi)} + \sum_{i=0}^{\infty} \psi_i \varepsilon_{t-i} \qquad (6)(6)$$

where

$$\psi_i = \begin{cases} 1, & if\ i = 0 \\ \phi^i + \theta\phi^{i-1}, & if\ i = 1, 2, 3, \dots \end{cases} \tag{7}$$

ARMA (1, 2)

$$y_t = c + \phi y_{t-1} + \varepsilon_t + \theta_1\varepsilon_{t-1} + \theta_2\varepsilon_{t-2} \tag{8}$$

$$y_t = \frac{c}{(1-\phi)} + \varepsilon_t + (\phi + \theta_1)\varepsilon_{t-1} + \sum_{i=2}^{\infty}(\phi^i + \theta_1\phi^{i-1} + \theta_2\phi^{i-2})\varepsilon_{t-i} \tag{9}$$

$$y_t = \frac{c}{(1-\phi)} + \sum_{i=0}^{\infty}\psi_i\varepsilon_{t-i} \tag{10}$$

where

$$\psi_i = \begin{cases} 1, & if\ i = 0 \\ \phi + \theta_1, & if\ i = 1 \\ \phi^i + \theta_1\phi^{i-1} + \theta_2\phi^{i-2}, & if\ i = 2, 3, 4, \dots. \end{cases} \tag{11}$$

ARMA (1, 3)

$$y_t = c + \phi y_{t-1} + \varepsilon_t + \theta_1\varepsilon_{t-1} + \theta_2\varepsilon_{t-2} + \theta_3\varepsilon_{t-3} \tag{12}$$

$$\begin{aligned} y_t = \frac{c}{(1-\phi)} &+ \varepsilon_t + (\phi + \theta_1)\varepsilon_{t-1} + (\phi^2 + \phi\theta_1 + \theta_2)\varepsilon_{t-2} \\ &+ \sum_{i=3}^{\infty}(\phi^i + \theta_1\phi^{i-1} + \theta_2\phi^{i-2} + \theta_3\phi^{i-3})\varepsilon_{t-i} \end{aligned} \tag{13}$$

$$y_t = \frac{c}{(1-\phi)} + \sum_{i=0}^{\infty}\psi_i\varepsilon_{t-i} \tag{14}$$

where

$$\psi_i = \begin{cases} 1, & i = 0 \\ \phi + \theta, & i = 1 \\ \phi^2 + \phi\theta_1 + \theta_2, & i = 2 \\ \phi^i + \theta_1\phi^{i-1} + \theta_2\phi^{i-2} + \theta_3\phi^{i-3}, & i = 3, 4, \dots \end{cases} \tag{15}$$

ARMA (1, 4)

$$y_t = c + \phi y_{t-1} + \varepsilon_t + \theta_1\varepsilon_{t-1} + \theta_2\varepsilon_{t-2} + \theta_3\varepsilon_{t-3} + \theta_4\varepsilon_{t-4} \tag{16}$$

$$\begin{aligned} y_t = \frac{c}{(1-\phi)} &+ \varepsilon_t + (\phi + \theta_1)\varepsilon_{t-1} + (\phi^2 + \phi\theta_1 + \theta_2)\varepsilon_{t-2} + (\phi^3 + \theta_1\phi^2 + \theta_2\phi + \theta_3)\varepsilon_{t-3} \\ &+ \sum_{i=4}^{\infty}(\phi^i + \theta_1\phi^{i-1} + \theta_2\phi^{i-2} + \theta_3\phi^{i-3} + \theta_4\phi^{i-4})\varepsilon_{t-i} \end{aligned} \tag{17}$$

$$y_t = \frac{c}{(1-\phi)} + \sum_{i=0}^{\infty}\psi_i\varepsilon_{t-i} \tag{18}$$

$$\psi_i = \begin{cases} 1, & i = 0 \\ \phi + \theta, & i = 1 \\ \phi^2 + \phi\theta_1 + \theta_2, & i = 2 \\ \phi^3 + \theta_1\phi^2 + \theta_2\phi + \theta_3, & i = 3 \\ \phi^i + \theta_1\phi^{i-1} + \theta_2\phi^{i-2} + \theta_3\phi^{i-3} + \theta_4\phi^{i-4}, & i = 4, 5, \dots \end{cases} \tag{19}$$

ARMA (1, q)

$$y_t = c + \phi y_{t-1} + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2} + \theta_3 \varepsilon_{t-3} + \theta_4 \varepsilon_{t-4} + \ \dots + \theta_q \varepsilon_{t-q} \qquad (20)$$

$$\begin{aligned} y_t = \frac{c}{(1-\phi)} &+ \varepsilon_t + (\phi + \theta_1)\varepsilon_{t-1} + (\phi^2 + \phi\theta_1 + \theta_2)\varepsilon_{t-2} + (\phi^3 + \theta_1\phi^2 + \theta_2\phi + \theta_3)\varepsilon_{t-3} \\ &+ (\phi^4 + \theta_1\phi^3 + \theta_2\phi^2 + \theta_3\phi + \theta_4)\varepsilon_{t-4} + \ \dots \\ &+ \sum_{i=q}^{\infty} (\phi^i + \theta_1\phi^{i-1} + \theta_2\phi^{i-2} + \theta_3\phi^{i-3} + \theta_4\phi^{i-4} + \ \dots + \theta_q\phi^{i-q})\varepsilon_{t-i} \end{aligned} \qquad (21)$$

$$y_t = \frac{c}{(1-\phi)} + \sum_{i=0}^{\infty} \psi_i \varepsilon_{t-i} \qquad (22)$$

where

$$\psi_i = \begin{cases} 1, & i = 0 \\ \phi + \theta, & i = 1 \\ \phi^2 + \phi\theta_1 + \theta_2, & i = 2 \\ \phi^3 + \theta_1\phi^2 + \theta_2\phi + \theta_3, & i = 3 \\ \phi^4 + \theta_1\phi^3 + \theta_2\phi^2 + \theta_3\phi + \theta_4, & i = 5 \\ \phi^i + \theta_1\phi^{i-1} + \theta_2\phi^{i-2} + \theta_3\phi^{i-3} + \theta_4\phi^{i-4} + \ \dots + \theta_q\phi^{i-q}, & i = q, q+1, \dots \end{cases} \qquad (23)$$

This establish linear process y_t as a linear combination of white noise variates ε_t. Equation (22) holds if $\phi \neq 1$ for all positive values of q for the case where constant value , c exists.

Invertibility of ARIMA model of various orders

ARIMA (p, 0, 1)

$$y_t = c + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \phi_3 y_{t-3} + \ \dots + \phi_p y_{t-p} + \varepsilon_t + \theta\varepsilon_{t-1} \qquad (24)$$

$$\begin{aligned} \varepsilon_t = -\frac{c}{(1+\theta)} &+ y_t - (\theta + \phi_1)y_{t-1} + (\theta^2 + \phi_1\theta - \phi_2)y_{t-2} - (\theta^3 + \phi_1\theta^2 - \phi_2\theta + \phi_3)y_{t-3} \\ &+ (\theta^4 + \phi_1\theta^3 - \phi_2\theta^2 + \phi_3\theta - \phi_4)y_{t-4} - \ \dots + \sum_{i=p}^{\infty} (-1)^i [\theta^i + \phi_1\theta^{i-1} - \phi_2\theta^{i-2} + \phi_3\theta^{i-3} - \phi_4\theta^{i-4} \\ &+ \ \dots + (-1)^{p+1}\phi_p\theta^{i-p}] y_{t-i} \end{aligned} \qquad (25)$$

$$\varepsilon_t = -\frac{c}{(1+\theta)} + \sum_{i=0}^{\infty} \pi_i y_{t-i} \qquad (26)$$

where

$$\pi_i = \begin{cases} 1, & i = 0 \\ -(\theta + \phi_1), & i = 1 \\ \theta^2 + \phi_1\theta - \phi_2, & i = 2 \\ -(\theta^3 + \phi_1\theta^2 + \phi_2\theta + \phi_3), & i = 3 \\ \theta^4 + \phi_1\theta^3 - \phi_2\theta^2 + \phi_3\theta - \phi_4, & i = 4 \\ (-1)^i [\theta^i + \phi_1\theta^{i-1} - \phi_2\theta^{i-2} + \phi_3\theta^{i-3} - \phi_4\theta^{i-4} + \\ \qquad \dots + (-1)^{p+1}\phi_p\theta^{i-p}], & i = p, p+1, \dots \end{cases} \qquad (27)$$

ARIMA (p, 1, 1)

$$\Delta y_t = c + \phi_1 \Delta y_{t-1} + \phi_2 \Delta y_{t-2} + \phi_3 \Delta y_{t-3} + \ \dots + \phi_p \Delta y_{t-p} + \varepsilon_t + \theta\varepsilon_{t-1} \qquad (28)$$

$$\begin{aligned} \varepsilon_t = -\frac{c}{(1+\theta)} &+ y_t - (\theta + \phi_1 + 1)y_{t-1} + [\theta^2 + \theta(1 + \phi_1) + \phi_1 - \phi_2]y_{t-2} \\ &- [\theta^3 + \theta^2(1 + \phi_1) + \theta(\phi_1 - \phi_2) - \phi_2 + \phi_3]y_{t-3} \\ &+ [\theta^4 + \theta^3(1 + \phi_1) + \theta^2(\phi_1 - \phi_2) - \theta(\phi_2 - \phi_3) + \phi_3 - \phi_4]y_{t-4} + \ \dots \\ &+ \sum_{i=p}^{\infty} (-1)^i [\theta^i + \theta^{i-1}(1 + \phi_1) + \theta^{i-2}(\phi_1 - \phi_2) - \theta^{i-3}(\phi_2 - \phi_3) + \theta^{i-4}(\phi_3 - \phi_4) + \cdots \\ &- \theta^{i-p}\phi_p] y_{t-i} \end{aligned} \qquad (29)$$

$$\varepsilon_t = -\frac{c}{(1+\theta)} + \sum_{i=0}^{\infty} \pi_i y_{t-i} \qquad (30)$$

Where

$$\pi_i = \begin{cases} 1, & i = 0 \\ -(\theta + \phi_1 + 1), & i = 1 \\ \theta^2 + \theta(1 + \phi_1) + \phi_1 - \phi_2, & i = 2 \\ -[\theta^3 + \theta^2(1 + \phi_1) + \theta(\phi_1 - \phi_2) - \phi_2 + \phi_3], & i = 3 \\ (-1)^i[\theta^i + \theta^{i-1}(1 + \phi_1) + \theta^{i-2}(\phi_1 - \phi_2) - \theta^{i-3}(\phi_2 - \phi_3) & \\ +\theta^{i-4}(\phi_3 - \phi_4) - \theta^{i-5}(\phi_4 - \phi_5) + \cdots + (-1)^{p+1}\theta^{i-p-1}\phi_p], i = p + 1, \dots & \end{cases} \quad (31)$$

The expression for ARIMA (p, d, 1) for various values of $p \geq 1$ holds if $\theta \neq -1$ for the case where constant term c exists but holds for all values of θ if the is no constant term.

Evaluation of parameters of ARIMA (p, d, q) given that $q = 0$

ARIMA (P, 0, 0)

$$y_t = c + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \phi_3 y_{t-3} + \dots + \phi_P y_{t-P} + \varepsilon_t \quad (32)$$

Given that $c = 0$, we have $A\boldsymbol{\Psi} = B$. That is

$$\begin{pmatrix} \sum_{t=1}^{T} {y_{t-1}}^2 & \sum_{t=1}^{T} y_{t-2}y_{t-1} & \sum_{t=1}^{T} y_{t-3}y_{t-1} & \cdots & \sum_{t=1}^{T} y_{t-p}y_{t-1} \\ & \vdots & & \vdots & \vdots \\ \sum_{t=1}^{T} y_{t-1}y_{t-p} & \sum_{t=1}^{T} y_{t-2}y_{t-p} & \sum_{t=1}^{T} y_{t-3}y_{t-p} & \cdots & \sum_{t=1}^{T} {y_{t-p}}^2 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_p \end{pmatrix} = \begin{pmatrix} \sum_{t=1}^{T} y_t y_{t-1} \\ \vdots \\ \sum_{t=1}^{T} y_t y_{t-p} \end{pmatrix}$$

where A is pxp matrix and $\boldsymbol{\Psi}$ is a column matrix, that is $\boldsymbol{\Psi} = (\phi_1 \ \ \phi_2 \ \ \phi_3 \ \dots \ \phi_p)'$**.** B is a column matrix, $\begin{pmatrix} \sum_{t=1}^{T} y_t y_{t-1} \\ \vdots \\ \sum_{t=1}^{T} y_t y_{t-p} \end{pmatrix}$.

The expression for each parameter $\phi_i, i = 1, 2, \dots, p$ can thus be determined using Crammer's rule or Gauss-Schidel method. But given that $c \neq 0$, we have $A\boldsymbol{\Psi} = B$, where $\boldsymbol{\Psi} = (\boldsymbol{c}, \phi_1 \ \ \phi_2 \ \ \phi_3 \ \dots \ \phi_p)'$.

$$\begin{pmatrix} n & \sum_{t=1}^{T} y_{t-1} & \sum_{t=1}^{T} y_{t-2} & \cdots & \sum_{t=1}^{T} y_{t-p} \\ & \vdots & & \vdots & \vdots \\ \sum_{t=1}^{T} y_{t-1}y_{t-p} & \sum_{t=1}^{T} y_{t-2}y_{t-p} & \sum_{t=1}^{T} y_{t-3}y_{t-p} & \cdots & \sum_{t=1}^{T} {y_{t-p}}^2 \end{pmatrix} \begin{pmatrix} c \\ \phi_1 \\ \vdots \\ \phi_p \end{pmatrix} = \begin{pmatrix} \sum_{t=1}^{T} y_t \\ \sum_{t=1}^{T} y_t y_{t-1} \\ \vdots \\ \sum_{t=1}^{T} y_t y_{t-p} \end{pmatrix}$$

ARIMA (P, 1, 0)

$$\Delta y_t = c + \phi_1 \Delta y_{t-1} + \phi_2 \Delta y_{t-2} + \phi_3 \Delta y_{t-3} + \dots + \phi_P \Delta y_{t-P} + \varepsilon_t \quad (33)$$

When $c = 0$, we have $A\boldsymbol{\Psi} = B$. That is

$$\begin{pmatrix} \sum_{t=1}^{T} (y_{t-1} - y_{t-2})^2 & \sum_{t=1}^{T} (y_{t-1} - y_{t-2})(y_{t-2} - y_{t-3}) & \cdots & \sum_{t=1}^{T} (y_{t-1} - y_{t-2})(y_{t-p} - y_{t-p-1}) \\ & \vdots & \vdots & \vdots \\ \sum_{t=1}^{T} (y_{t-1} - y_{t-2})(y_{t-p} - y_{t-p-1}) & \sum_{t=1}^{T} (y_{t-2} - y_{t-3})(y_{t-p} - y_{t-p-1}) & \cdots & \sum_{t=1}^{T} (y_{t-p} - y_{t-p-1})^2 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_p \end{pmatrix}$$

$$= \begin{pmatrix} \sum_{t=1}^{T} (y_t - y_{t-1})(y_{t-1} - y_{t-2}) \\ \vdots \\ \sum_{t=1}^{T} (y_t - y_{t-1})(y_{t-p} - y_{t-p-1}) \end{pmatrix}$$

When $c \neq 0$, we have

$$\begin{pmatrix} \sum_{t=1}^{T}(y_{t-1}-y_{t-2}) & \sum_{t=1}^{T}(y_{t-1}-y_{t-2})^2 & \cdots & \sum_{t=1}^{T}(y_{t-1}-y_{t-2})(y_{t-p}-y_{t-p-1}) \\ \sum_{t=1}^{T}(y_{t-2}-y_{t-3}) & \sum_{t=1}^{T}(y_{t-1}-y_{t-2})(y_{t-2}-y_{t-3}) & & \sum_{t=1}^{T}(y_{t-1}-y_{t-2})(y_{t-p}-y_{t-p-1}) \\ \vdots & \vdots & \vdots & \vdots \\ \sum_{t=1}^{T}(y_{t-p}-y_{t-p-1}) & \sum_{t=1}^{T}(y_{t-1}-y_{t-2})(y_{t-p}-y_{t-p-1}) & \cdots & \sum_{t=1}^{T}(y_{t-p}-y_{t-p-1})^2 \end{pmatrix}\begin{pmatrix} c \\ \phi_1 \\ \vdots \\ \phi_p \end{pmatrix}$$

$$= \begin{pmatrix} \sum_{t=1}^{T}(y_t-y_{t-1})(y_{t-1}-y_{t-2}) \\ \sum_{t=1}^{T}(y_t-y_{t-1})(y_{t-2}-y_{t-3}) \\ \vdots \\ \sum_{t=1}^{T}(y_t-y_{t-1})(y_{t-p}-y_{t-p-1}) \end{pmatrix}$$

For the estimate of parameters in ARIMA (p, 0, 0) and (p, 1, 0), it is deduced that every term y_{t-j} in ARIMA (p, 0, 0) is replaced by $y_{t-j} - y_{t-j-1}$ in ARIMA (p, 1, 0). Also, $\boldsymbol{\Psi}$ is p column matrix for $c = 0$ while $\boldsymbol{\Psi}$ is $(p + 1)$ column matrix for $c \neq 0$.

Behavioural pattern of causality parameters of ARMA (*1*, *q*) model

Given that $\theta_k > \theta_{k+1} > \theta_{k+2} > \cdots > \theta_{k+q} > 0$ for various values of q, the causality parameter ψ_i of ARMA (1, *q*) is skewed to right for positive values of ϕ and sinusoidal for negative values of ϕ as shown in Figures 1 – 3 below. The absolute value of causality parameter $|\psi_i|$ rises as *d* increases for positive values of ϕ but smaller the value of *d*, the bigger the absolute value of causality parameter ψ_i for negative values of ϕ. Also, ψ_i increases as ϕ increases as shown in Figure 4 for positive values of ϕ. Similarly, the invertibility parameter π_i of ARIMA (p, d, 1) where $d = 0$ and 1 is sinosidal and that $|\pi_i|_{d=1} > |\pi_i|_{d=0}$ for positive values of θ. Also, $|\pi_i|_{d=0}$ converges faster to zero than $|\pi_i|_{d=1}$ for positive values of θ as can be seen in Figure 5 below. The parameters of ARIMA were evaluated for various values of p and c at $q = 0$ using ordinary least squares method and Crammer's rule.

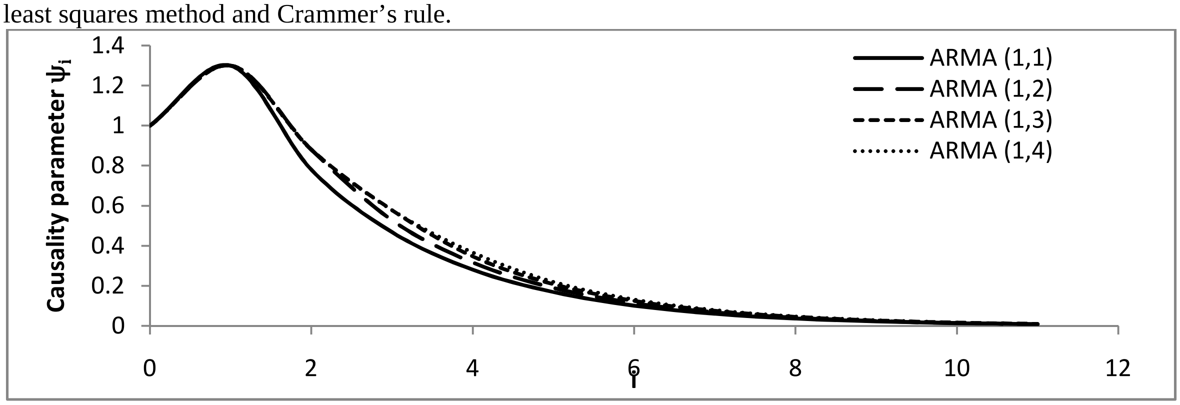

Figure 1: Figure showing behaviour of causality parameter ψ_i of ARMA (*1*, *q*) for some *i* given that $\phi = 0.6$, $\theta_1 = 0.7$, $\theta_2 = 0.1$, $\theta_3 = 0.05$ and $\theta_4 = 0.02$.

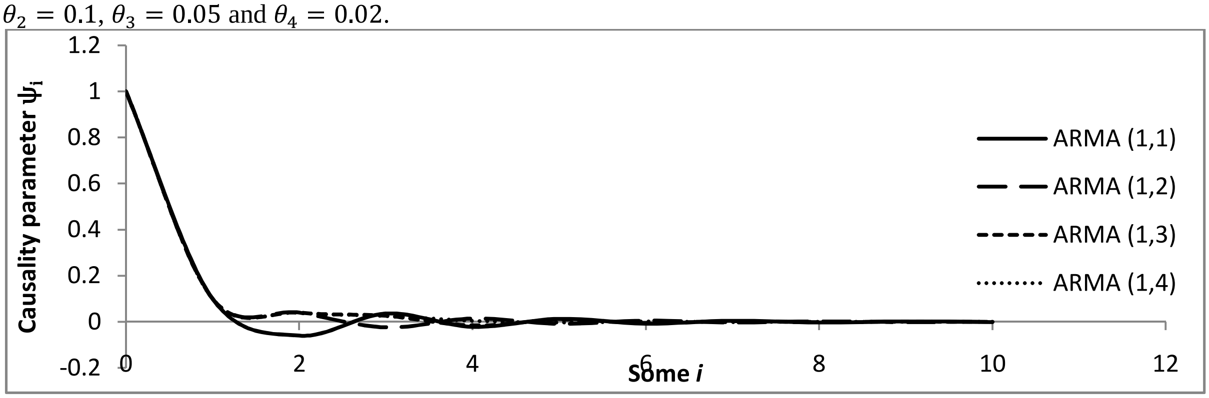

Figure 2: Figure showing behaviour of ψ_i of ARMA (*1*, *q*) for some *i* given that $\phi = -0.6$, $\theta_1 = 0.7$, $\theta_2 = 0.1$, $\theta_3 = 0.05$ and $\theta_4 = 0.02$.

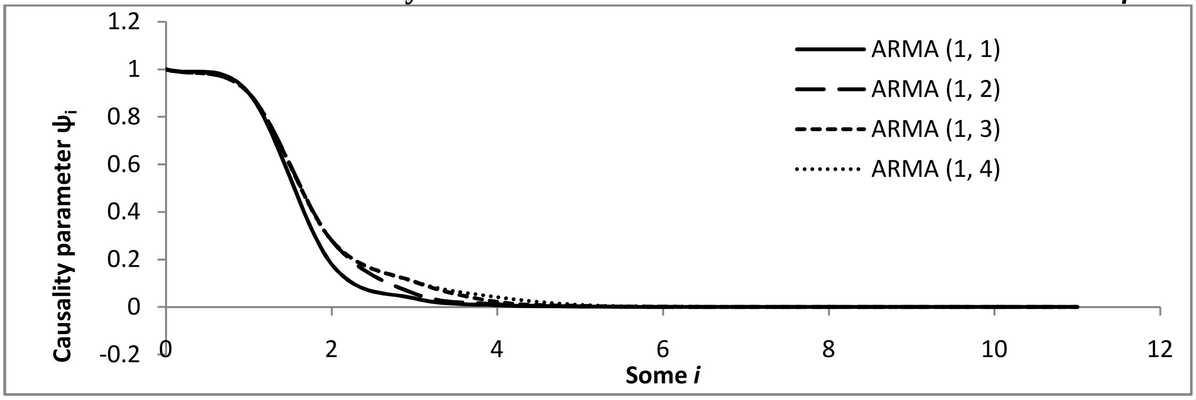

Figure 3: Figure showing behaviour of ψ_i of ARMA (*1, q*) for some *i* given that $\phi = 0.2$, $\theta_1 = 0.7$, $\theta_2 = 0.1$, $\theta_3 = 0.05$ and $\theta_4 = 0.02$.

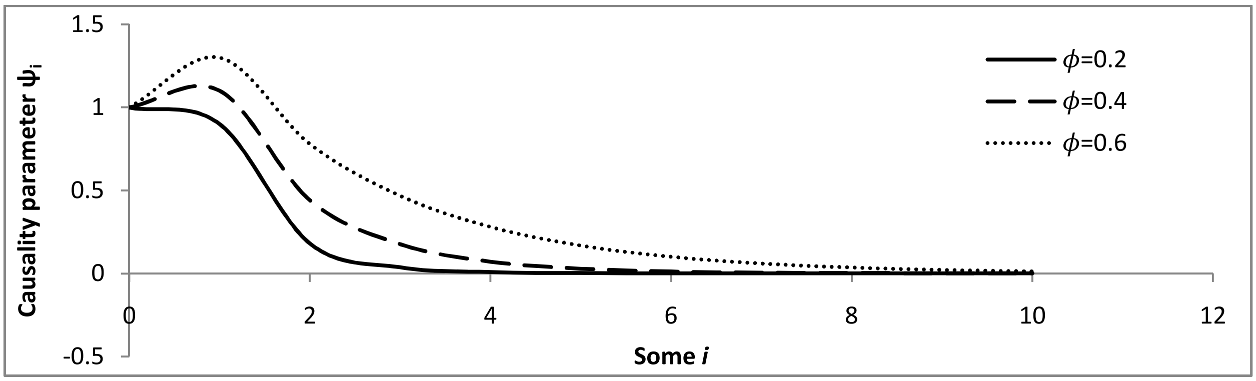

Figure 4: Figure showing behaviour of ψ_i of ARMA (*1, q*) for some *i* given that $\theta_1 = 0.7$, $\theta_2 = 0.1$, $\theta_3 = 0.05$ and $\theta_4 = 0.02$ for some real values of ϕ.

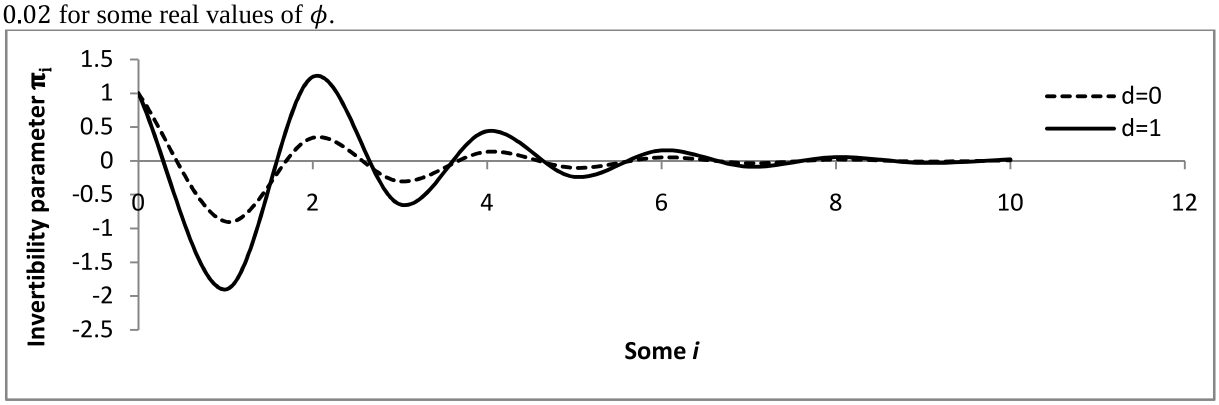

Figure 5: Graph showing behaviour of invertibility parameter π_i of ARIMA (p, d, 1) when $d = 0$ and $d = 1$ given that $\theta = 0.6$, $\phi_1 = 0.3$, $\phi_2 = 0.2$, $\phi_3 = 0.1$, $\phi_4 = 0.05$, $\phi_5 = 0.025$, $\phi_6 = 0.0125$, $\phi_7 = 0.00625$, $\phi_8 = 0.003125$, $\phi_9 = 0.0015625$ and $\phi_{10} = 0.00078125$.

4. Conclusion

In this paper, ARMA model of various orders was presented in causal forms. The result in equation (26) confines [8]. It was deduced that behaviour of causality parameter ψ_i depends on positive and negative values of ϕ and θ. Causality parameter ψ_i is skewed to the right and sinusoidal for positive and negative values of ϕ respectively. Absolute value of causality parameter ψ_i of ARIMA (1, 0, *q*) increases as the value of q increases for positive values of ϕ. Similarly, the invertibility parameter π_i of ARIMA (p, d, 1) where $d = 0$ and 1 is sinosidal and that $|\pi_i|_{d=1} > |\pi_i|_{d=0}$ for positive values of θ. Also, $|\pi_i|_{d=0}$ converges faster to zero than $|\pi_i|_{d=1}$ for positive values of θ as can be seen in Figure 5 above. The parameters of ARIMA were evaluated for various values of p and c at $q = 0$ using ordinary least squares method and Crammer's rule.

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