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A New Formulation for Symmetric Implicit Runge-Kutta-Nystrom Method for the Integration of General Second Order ODEs

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Abstract

Symmetric Runge-Kutta –Nystrom Methods are of much current interest due to their efficiency when solving stiff systems and the possibility to use them as basic methods for extrapolation processes.In this paper we derive symmetric stable Implicit Runge-Kutta –Nystrom Method for the Integration of General Second Order ODEs by using the collocation approach.The block hybrid method obtained by the evaluation of the continuous interpolant at different nodes of the polynomial is symmetric and suitable for stiff intial value ODEs problems. We further coverted the uniformly accurate order six block hybrid method to symmetric Implicit Runge-Kutta –Nystrom Method by using the direct method as those invented by Nystrom.The convergence of the method is achieved as shown in the table of results.

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1.0 Introduction

There is a vast body of literature addressing the numerical solution of the so called special second order initial value problems(IVP).

$$y'' = f(x, y) \qquad y(x_o) = y \qquad y'(x_o) = \beta \tag{1.1}$$

(see for example[1] and[2] but not so much for the general second order IVP with a dissipative term

$$y'' = f(x, y, y') \qquad y(x_o) = y \qquad y'(x_o) = \beta \tag{1.2}$$

(Different approaches appear in [3],[4],[5] and [6]).

Although it is possible to integrate a second order IVP by reducing it to first order system and apply one of the method available for such system it seem more natural to provide commercial method in order to integrate the problem directly. The advantage of these approaches lies in the fact that they are able to exploit special information about ODEs and this result in an increase in efficiency (that is ,high accuracy at low cost) For instance ,it is well know that Runge-kutta Nystrom method for (1.2) involve a real improvement as compared to standard Runge-kutta method for a given number of stages [7,p.285].

In this paper,we present a five stage implicit Runge-kutta Nystrom method for direct integration of second order ODEs with the following advantage such as high order and stage order, low error constant and low implementation cost.

An s-stage implicit Runge-kutta Nystrom for direct integration of general second order IVP (1.2) is defined in the form

$$y_{n+1} = y_n + \alpha_i h y'_n + h^2 \sum_{j=1}^{i-1} a_{ij} k_j \tag{1.3}$$

$$y'_{n+1} = y'_n + h \sum_{j=1}^{i-1} \bar{a}_{ij} k_j \tag{1.4}$$

Where for i = 1, 2 - - - - s.

$$K_i = f\left(x_i + \alpha_j h,\ y_n + \alpha_i y'_n + h^2 \sum_{j=1}^{i-1} a_{ij} k_j,\ y'_n + h \sum_{j=1}^{i-1} \bar{a}_{ij} k_j \right) \tag{1.5}$$

The real parameters $\alpha_j, k_i, a_{ij}, \bar{a}_{ij}$ define the method.

The paper is organized as follows, in section 2 we will show how the butcher's implicit Runge-kutta methods for the first

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order differential equations tableau are modified to include second derivative(that is implicit Runge-kutta Nystrom method) that will be used later in section 4, in section 3 we offer the main construction of the symmetric Implicit Runge-Kutta – Nystrom Method,in section 4 we develop a procedure to obtain the implementation of the symmetric Implicit Runge-Kutta – Nystrom Method and describes how to obtain the coefficient in section 2 finally, some numerical experiments are presented in section 5.

2. Butcher's implicit Runge-kutta methods for the first order differential equations

For the first order differential equations

$$y' = f(x, y) \qquad y(x_o) = y \tag{1.6}$$

[4] and [8] defined an s-stage implicit Runge-kutta methods in the form

$$y_{n+1} = y_n + h\sum_{i=1}^{s} w_i k_i \tag{1.7}$$

Where for $i = 1, 2 - - - -$ s.

$$K_i = f(x_i + \alpha_j h,\ y_n + h\sum_{j=1}^{s} a_{ij} k_j) \tag{1.8}$$

The real parameters α_j, k_i, a_{ij} define the method. The method (1.7) in Butcher-array form can be written as

$$\begin{array}{c|c} \alpha & \beta \\ \hline & W^T \end{array} \tag{1.9}$$

While for the implicit Runge–Kutta method for the numerical integration of the second order initial value problem (1.1and 1.2) the method (1.3) in Butcher – array form.

$$\begin{array}{c|c|c} \alpha & \bar{A} & A \\ \hline & \bar{b}^T & b \end{array}$$

$$A = a_{ij} = \beta^2 \qquad \bar{A} = \bar{a}_{ij} = \beta$$

$$\beta = \beta e \qquad \bar{b} = W \qquad b = W^T\beta \tag{2.0}$$

(see Hairer and Wanner[9)

3.0 Construction Of The Method

We particularly wish to emphasize the combination of a multi-step structure with the use of off-step points, we seek a method that are multistage and multi-value because it will be convenient to extend the general linear method formulation to the high order Runge – Kutta case Butcher and Wright [4] by considering a polynomial.

$$y(x) = \sum_{j=1}^{i-1} \phi_j(x) y_{n+j} + h\sum_{j=1}^{i-1} \varphi_j(x) f(\bar{x}_j, y(\bar{x}_j)) \tag{2.1}$$

Where t donate the number of interpolation point x_{n+j}, j =0,1--- *t-1*; and m donates the distinct collocation points $\bar{x}_j \in [x_n, x_{n+k}]$,j = 0,1---m-1 chosen from the given step $[x_n, x_{n+k}]$. Here *y* and *f* are smooth real N-dimensional vector functions. The numerical constant coefficient ϕ_j,(j = 0,1---t-1) and $h\,\phi_j$, (j = 0,1---m-1) of the matrix C in(2.9) below are to be determined since the matrix has strictly (t+m)*(t+m) dimension (square matrix) they are selected so that accurate approximations of well behaved problems step size can be a constant or change in the numerical integration process. In (2.0) A donate the i(t+m)*(t+m) real matrix and b and α are real vectors of dimension (t+m) and $\alpha_i \in$ [0, k], ί = 1, 2 --- t+m-1.According to Kulikov [10] if the matrix A in the butcher's array is a lower triangular matrix with zero main diagonal, then the method is called explicit.The function $\phi_j(x)$ and $\varphi_j(x)$ can be represented by polynomials of the form.

$$\begin{aligned} \phi_j(x) &= \sum_{i=o}^{t+m-1} \phi_{j,i+1} x^j,\ j \in \{0,1,---,t-1\} \\ h\varphi_j(x) &= \sum_{i=o}^{t+m-1} \varphi_{j,i+1} x^j,\ j \in \{0,1,---,m-1\} \end{aligned} \tag{2.3}$$

With constant coefficients $\phi_{j,i+1}$ and $h\varphi_{j,i+1}$ i to be determined. Putting (2.3) back into (2.1) we have

$$y(x) = \sum_{j=o}^{t-1} \sum_{i=o}^{t+m-1} \phi_{j,i+1} x^i y_{n+j} + h \sum_{i=o}^{m-1} \sum_{i=o}^{t+m-1} \varphi_{j,i+1} x^i f_{n+j}$$

$$= \sum_{i=o}^{t+m-1} \left\{ \sum_{i=o}^{t-1} \phi_{j.i+1} y_{n+j} + \sum_{j=o}^{m-1} h\varphi_{j,i+1} f_{n+j} \right\} x^i \qquad (2.4)$$

Writing

$$a_i = \sum_{j=o}^{t-1} \phi_{j,i+1} y_{n+j} + \sum_{j=o}^{m-1} h\varphi_{j.i+1} f_{n+j}$$

Such that (2.4) reduce to a power series of a single variable x in the form

$$P(x) = \sum_{j=o}^{\infty} a_j x^i \qquad (2.5)$$

And is used as the basis or trial function to produce an approximate solution to (1.1),(1.2) and (1.6) as

$$y(x) = \sum_{j=o}^{t+m-1} a_j x^j$$

$$a; \in R,\ j = o(1)t + m - 1, Y \in C^m(a,b) \subset P(x) \qquad (2.6)$$

which can be express as

$$y(x) = \left(\sum_{j=0}^{t-1} \Phi_j, t + m - 1 y_{n+j} + \sum_{j=0}^{m-1} h\psi_{j.t+m} - f_{n+j} \right) \left(1, x, x^2 ----x^{t+m-1} \right)^T \qquad (2.7)$$

Thus we can express equation (2.7) explicitly as follows

$$y(x) = [y_n \cdots y_{n+t-1,}\ fn \cdots fn + m - 1] C^T \begin{bmatrix} 1 \\ x \\ x^2 \\ --- \\ x^{t+m-1} \end{bmatrix} \qquad (2.8)$$

$$C = \begin{bmatrix} \Phi_{o,0} & \cdots & \Phi_{t-1,1} & h\Psi_{0,1} & \cdots & h\Psi_{m-1,1} \\ \Phi_{o,2} & \cdots & \Phi_{t-1,2} & h\Psi_{0,2} & \cdots & h\Psi_{m-1,2} \\ \Phi_{o,3} & \cdots & \Phi_{t-1,3} & h\Psi_{0,3} & \cdots & h\Psi_{m-1,3} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \Phi_{o,t+m} & & \Phi_{t-1,t+m} & h\Psi_{0,t+m} & \cdots & h\Psi_{m-1,t+m} \end{bmatrix} = D^{-1} \qquad (2.9)$$

of dimension (t+m) * (t+m) and

$$D = \begin{bmatrix} 1 & x_n & x^2{}_n & \cdots & x_n^{t+m-1} \\ 1 & x_{n+1} & x^2{}_{n+1} & \cdots & x_{n+1}^{t+m-1} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & x_{n+t-1} & x^2{}_{n+1} & \ddots & x_{n+t-1}^{t+m-1} \\ 0 & 1 & 2\overline{x}_0 & \cdots & (t+m-1)\overline{x}^{t+m-2} \\ \vdots & \vdots & \vdots & & \\ 0 & 1 & 2\overline{x}_{m-1} & \cdots & (t+m-1)\overline{x}_{m-1}^{t+m-2} \end{bmatrix}. \qquad (3.0)$$

We call D the multistep collocation matrix which has a very simple structure and dimension (t+m) x (t+m). as can be seen the entries of C are the constant coefficients of the Polynomials given in (2.6) which are to be determined.

Following the multistep collocation of Onumanyi etal[11],We invert once the matrix D of the form in (3.0) which is of dimension (t+m)*(t+m) where m=t 3,both at point $\frac{1}{4},\frac{1}{2}$ and $\frac{3}{4}$, $\phi=\alpha$ and $\varphi=\beta$ with the continuous scheme as

$$y(x)=\frac{\left[-19h^5+114h^4(x-x_n)+254h^3(x-x_n)^2+262h^2(x-x_n)^3-128h(x-x_n)^4+24(x-x_n)^5\right]}{h^5}y_{n+\frac{1}{4}}$$
$$+\frac{\left[9h^5-48h^4(x-x_n)+88h^3(x-x_n)^2-64h^2(x-x_n)^3+16h(x-x_n)^4\right]}{h^5}y_{n+\frac{1}{2}}$$
$$+\frac{\left[10h^5-66h^4(x-x_n)+166h^3(x-x_n)^2-196h^2(x-x_n)^3+112h(x-x_n)^4-24(x-x_n)^5\right]}{h^5}y_{n+\frac{3}{4}}$$
$$+\frac{\left[-\frac{9}{2}h^5+24h^4(x-x_n)-97h^3(x-x_n)^2+47h^2(x-x_n)^3-22h(x-x_n)^4+4(x-x_n)^5\right]}{h^4}f_{n+\frac{1}{4}}$$
$$+\frac{\left[-9h^5+57h^4(x-x_n)-136h^3(x-x_n)^2+156h^2(x-x_n)^3-80h(x-x_n)^4+16(x-x_n)^5\right]}{h^4}f_{n+\frac{1}{2}}$$
$$+\frac{\left[-\frac{3}{2}h^5+10h^4(x-x_n)-\frac{51}{2}h^3(x-x_n)^2+31h^2(x-x_n)^3-18h(x-x_n)^4+4(x-x_n)^5\right]}{h^4}f_{n+\frac{3}{4}} \quad (3.1)$$

Evaluating equation (3.1) and its first derivative at points x_n and x_{n+1} we obtained the following block hybrid schemes with uniformly accurate order six and error constant $\left(\frac{-439}{1935360},\frac{-17}{120960},\frac{-19}{71680},\frac{1}{15120}\right)^T$.

$$18y_{n+\frac{1}{4}}-9y_{n+\frac{1}{2}}-10y_{n+\frac{3}{4}}+y_n=\frac{h}{4}\left\{-9f_{n+\frac{1}{4}}-18f_{n+\frac{1}{2}}-3f_{n+\frac{3}{4}}\right\}$$
$$-10y_{n+\frac{1}{4}}-9y_{n+\frac{1}{2}}+18y_{n+\frac{3}{4}}+y_{n+1}=\frac{h}{4}\left\{3f_{n+\frac{1}{2}}+18f_{n+\frac{1}{2}}+9f_{n+\frac{3}{4}}\right\}$$
$$-228y_{n+\frac{1}{4}}+96y_{n+\frac{1}{2}}+132y_{n+\frac{3}{4}}=h\left\{24f_{n+\frac{1}{4}}+57f_{n+\frac{1}{2}}+10f_{n+\frac{3}{4}}-f_n\right\}$$
$$-132y_{n+\frac{1}{4}}-96y_{n+\frac{1}{2}}+228y_{n+\frac{3}{4}}=h\left\{10f_{n+\frac{1}{4}}+57f_{n+\frac{1}{2}}+24f_{n+\frac{3}{4}}-f_{n+1}\right\} \quad (3.2)$$

4.0 Implementation Of The Symmetric Implicit Runge-Kutta –Nystrom Method

Solving the block implicit hybrid scheme simultaneously we obtained the following block scheme

$$y_{n+\frac{1}{4}}=y_n+\frac{h}{2880}\left\{251f_n+646f_{n+\frac{1}{4}}-264f_{n+\frac{1}{2}}+106f_{n+\frac{3}{4}}-19f_1\right\}$$
$$y_{n+\frac{1}{2}}=y_n+\frac{h}{360}\left\{29f_n+124f_{n+\frac{1}{4}}+24f_{n+\frac{1}{2}}+4f_{n+\frac{3}{4}}-f_1\right\}$$
$$y_{n+\frac{3}{4}}=y_n+\frac{h}{320}\left\{27f_n+102f_{n+\frac{1}{4}}+72f_{n+\frac{1}{2}}+42f_{n+\frac{3}{4}}-3f_1\right\}$$

$$y_{n+1} = y_n + \frac{h}{90}\left\{7f_n + 32f_{n+\frac{1}{4}} + 12f_{n+\frac{1}{2}} + 32f_{n+\frac{3}{4}} + 7f_1\right\} \tag{3.3}$$

Reformulating the block hybrid method in the general linear method for first and second derivative (see Butcher and Wright[4] and Chollom and Jackiewicz[12]) and the coefficients as characterized by the butcher array(1.4) and (1.5) we obtain respectively

$$\begin{array}{c|ccccc}
0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{4} & \frac{251}{2880} & \frac{323}{1440} & \frac{-11}{120} & \frac{53}{1440} & \frac{-19}{2880} \\
\frac{1}{2} & \frac{29}{360} & \frac{31}{90} & \frac{1}{15} & \frac{1}{90} & \frac{-1}{360} \\
\frac{3}{4} & \frac{27}{320} & \frac{51}{160} & \frac{9}{40} & \frac{21}{160} & \frac{-3}{320} \\
1 & \frac{7}{90} & \frac{16}{45} & \frac{2}{15} & \frac{16}{45} & \frac{7}{90} \\
\hline
 & \frac{7}{90} & \frac{16}{45} & \frac{2}{15} & \frac{16}{45} & \frac{7}{90}
\end{array} \tag{3.4}$$

$$\begin{array}{c|ccccc|ccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{4} & \frac{251}{2880} & \frac{323}{1440} & \frac{-11}{120} & \frac{53}{1440} & \frac{-19}{2880} & \frac{17}{1152} & \frac{9}{320} & \frac{-37}{1920} & \frac{7}{720} & \frac{-1}{480} \\
\frac{1}{2} & \frac{29}{360} & \frac{31}{90} & \frac{1}{15} & \frac{1}{90} & \frac{-1}{360} & \frac{13}{360} & \frac{37}{360} & -\frac{1}{40} & \frac{1}{72} & \frac{-1}{360} \\
\frac{3}{4} & \frac{27}{320} & \frac{51}{160} & \frac{9}{40} & \frac{21}{160} & \frac{-3}{320} & \frac{9}{160} & \frac{3}{16} & \frac{9}{640} & \frac{9}{320} & \frac{-3}{640} \\
1 & \frac{7}{90} & \frac{16}{45} & \frac{2}{15} & \frac{16}{45} & \frac{7}{90} & \frac{7}{90} & \frac{4}{15} & \frac{1}{15} & \frac{4}{45} & 0 \\
 & \frac{7}{90} & \frac{16}{45} & \frac{2}{15} & \frac{16}{45} & \frac{7}{90} & \frac{7}{90} & \frac{4}{15} & \frac{1}{15} & \frac{4}{45} & 0 \\
\hline
\end{array} \tag{3.5}$$

As characterized by the theory of Nystôm method (see Haire and Wanner[8] and Butcher and Hojjat[13]).
An s-stage implicit Runge – Kutta method for the direct integration of second order initial value problem (1.1) and (1.2) is determined in the form

$$y_{n+1} = y_n + \alpha_i h y'_n + h^2 \sum_{j=1}^{i-1} a_{ij} k_j \qquad \ldots(3.6)$$

$$y'_{n+1} = y'_n + h \sum_{j=1}^{i-1} \bar{a}_{ij} k_j \qquad \ldots(3.7)$$

Where for i = 1, 2 - - - - s.

$$K_i = f\left(x_i + \alpha_j h,\ y_n + \alpha_i y'_n + h^2 \sum_{j=1}^{i-1} a_{ij} k_j,\ y'_n + h \sum_{j=1}^{i-1} \bar{a}_{ij} k_j\right) \qquad \ldots(3.8)$$

The real parameters $\alpha_j, k_i, a_{ij}, \overline{a}_{ij}$ define the method.

Using equation (3.6) we obtained a symmetric implicit Runge-Kutta –Nystrom method of uniform order six everywhere on the interval of solution Yakub et al[14] and chollom and Jackiewicz[12].

$$
\begin{aligned}
k_1 &= f(x_n, y_n, y_n') \\
k_2 &= f(x_n + \tfrac{1}{4}h, y_n + \tfrac{1}{4}hy_n' + h^2(\tfrac{17}{1152}k_1 + \tfrac{9}{320}k_2 - \tfrac{37}{1920}k_3 + \tfrac{7}{720}k_4 - \tfrac{1}{480}k_5), \\
&\quad y_n' + h(\tfrac{251}{2880}k_1 + \tfrac{323}{1440}k_2 - \tfrac{11}{120}k_3 + \tfrac{53}{1440}k_4 - \tfrac{19}{2880}k_5)) \\
k_3 &= f(x_n + \tfrac{1}{2}h, y_n + \tfrac{1}{2}hy_n' + h^2(\tfrac{13}{360}k_1 + \tfrac{37}{360}k_2 - \tfrac{1}{40}k_3 + \tfrac{1}{72}k_4 - \tfrac{1}{360}k_5), \\
&\quad y_n' + h(\tfrac{29}{360}k_1 + \tfrac{31}{90}k_2 + \tfrac{1}{15}k_3 + \tfrac{1}{90}k_4 - \tfrac{1}{360}k_5)) \\
k_4 &= f(x_n + \tfrac{3}{4}h, y_n + \tfrac{3}{4}hy_n' + h^2(\tfrac{9}{160}k_1 + \tfrac{3}{16}k_2 + \tfrac{9}{640}k_3 + \tfrac{9}{320}k_4 - \tfrac{3}{640}k_5), \\
&\quad y_n' + h(\tfrac{27}{320}k_1 + \tfrac{51}{160}k_2 + \tfrac{9}{40}k_3 + \tfrac{21}{160}k_4 - \tfrac{3}{320}k_5)) \\
k_5 &= f(x_n + h, y_n + hy_n' + h^2(\tfrac{7}{90}k_1 + \tfrac{4}{15}k_2 + \tfrac{1}{15}k_3 + \tfrac{4}{45}k_4 + 0k_5), \\
&\quad y_n' + h(\tfrac{7}{90}k_1 + \tfrac{16}{45}k_2 + \tfrac{2}{15}k_3 + \tfrac{16}{45}k_4 + \tfrac{7}{90}k_5)) \\
y_{n+1} &= y_n + hy_n' + h^2(\tfrac{7}{90}k_1 + \tfrac{4}{15}k_2 + \tfrac{1}{15}k_3 + \tfrac{4}{45}k_4 + 0k_5) \\
y_{n+1}' &= y_n' + h(\tfrac{7}{90}k_1 + \tfrac{16}{45}k_2 + \tfrac{2}{15}k_3 + \tfrac{16}{45}k_4 + \tfrac{7}{90}k_5)
\end{aligned}
\tag{3.9}
$$

5.0 Numerical Experiment

Problem 1

Dalquist and Bjock[15] showed that there are some stiff problems for which Runge- Kutta(R-K) method is unsuitable. One of such problem is the second order stiff differential equation

$$y'' = 1001y' + 1000y \qquad y(0) = 1 \qquad y'(0) = -1$$

This problem has a general solution given by

$$Y(x) = Ae^{-x} + Be^{-1000x}$$

The exact solution is given $y(x) = e^{-x}$

Sekar et al [16] gave some notes on how R-K Butcher algorithm could be used for this system. By setting $y' = z$, equation (4.1) becomes a system of stiff initial value problem hence, we have

$$y' = z \qquad y(0) = 1$$
$$z' = -1001z - 1000y \qquad z(0) = -1$$

This can be written as

$$\underline{Y'} = AY, \qquad \underline{Y} = (y.z)^T \text{ and } A = \begin{pmatrix} 0 & 1 \\ -1000 & -1001 \end{pmatrix}$$

The eigenvalues of the coefficient matrix A of equation (4.1) are $\lambda_1 = -1$ and $\lambda_2 = -1000$. By using the explicit fourth order R-K method, Dalquist and Bjock[15] showed that the method explodes for a step length h > 0.0025. A similar argument is adduced by Shampine and Gladwell [17] despite that this is unsatisfactory step size for describing the function e^{-x}.

On the contrary, this same problem is solved using our proposed new method with step-lengths greater than 0.0025.

The results obtained at x =1 using both the R-K method and the New Symmetric Implicit Runge-Kutta –Nystrom Method (NSIRKN) are given in table 1 below.

Table 1: Absolute errors of Problem 1

H	R-K	NSIRKN
0.0025	5.6E-06	2.5E-12
0.05	4.6E-03	2.8E-11
0.1	Explodes	2.7E-10

It will be observed that the explicit R-K method could not cope with this problem h>0.0025. The newly proposed NSIRKN gave a far better accuracy than the explicit R-K scheme.

Problem 2

$$y'' + 8y' + ky = 0 \qquad y(0) = 1 \qquad y'(0) = 12 \qquad k = 16$$

Theoretical Solution**:** $y(x) = (1-8x)e^{-4x}$

Source:www.faculty.valencia.cc.fi.us/pfernandz/des

/chapter5.pdf (20.9.2006) [2].

The notations used in the table below are as follows:MTHD - Method used, H - The size of the step ,FCN - the number of functions evaluations, STEP - the number of steps, ERR - max (absolute value of the true solution minus the computed solution at the mesh point *i*), SDIRKNG - A diagonally implicit Runge-Kutta-Nystróm General method(FUDZIAH[2]) and NSIRKN - New Symmetric Implicit Runge-Kutta –Nystrom Method (The newly proposed method) and 0.1234(-10) means $0.1234X10^{-10}$

Table 2: Absolute errors of Problem 2

MTHD	H	FCN	STEP	ERR
NSIRKN SDIRKNG	0.1	1100 1100	100 100	1.4583(-8) 3.8330(-3)
NSIRKN SDIRKNG	0.01	11000 11000	1000 1000	1.0174(-9) 3.1762(-6)
NSIRKN SDIRKNG	0.001	110000 110000	10000 10000	1.2876(-10) 3.1140(-8)

Problem 3 $y'' - y' = 0$, $y(0) = 0$, $y'(0) = -1$, h=0.1, $0 \le x \le 0.4$

Theoretical Solution**:** $y(x) = 1 - e^{x}$

Table 3: Absolute errors of Problem 3

X	LMM(YAHAYA [18])	NSIRKN
0.1	9.E-05	2.E+11
0.2	3.E-03	1.E-10
0.3	2.E-03	1.E-10
0.4	5.E-05	2.E-10

5. **Conclusion**

The approach presented in this paper, we can give the error constants and the continuous form is also available for dense approximation to the solution of a general second order ordinary differential equations.The method requires less work with very little cost (when compared with classical RK) and possesses a gain in efficiency (when compared with improved R-K Nyström and LMM); the method is self starting with no overlapping of solution models.

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