Journal of the Nigerian Association of Mathematical Physics
***Volume* 19 (November, 2011), pp 387 – 394**
© J. of NAMP

Analysis of Diffusion Process in an Infinite Core

Adeniji, A. A.

Department Of Petroleum Engineering,
University Of Benin, Nigeria.

Abstract

Mathematical and graphical illustrations of unsteady state diffusion in infinite porous core are analysed here for proper reservoir characterization. Since petroleum engineers are always advocating for some types of mathematical theory necessary to the application of laboratory data to field problems, the solution to one direction diffusivity equation based upon superposition of exponential and error-function solutions is presented. The analysis is valuable in estimating near wellbore effects in field well test analysis.

1.0 Introduction

The governing equation for flow of fluid within the reservoir rock is diffusivity equation defined in terms of pressure. Traditional core plug measurement is based on displacement process with Darcy's equation being the guiding principle and is based on steady flow in linear core. The core plug experiments are suitable for evaluating permeability in relatively homogeneous reservoirs, if there is adequate sampling. In these reservoirs, the arithmetic average of the core plug data is taken to be effective upscaling of reservoir permeability. Very few reservoirs have uniform properties and the core taken from bore hole cannot represent the whole formation under investigation. To even have a resemblance of reservoir characteristics the core measurement must be frequent enough to capture the reservoir heterogeneities. The high permeability and the low permeability zones are likely to be missed. Thus such traditional sampling scheme gives a significant difference or biased average.

In field, well test analysis of pressure profile always provide estimate of reservoir permeability. In heterogeneous, reservoirs core data are average for comparison with well test data based on diffusivity equation. Averaging is commonly used as a form of upscaling and it is the function of the analyst to find the best upscaling method to fit the data. Jensen et al., (1997) proposed power average technique for aggregating the core permeability's' upscaling such that well test permeability is a power average of permeability variation in the average volume of investigation. Oliver (1990) earlier demonstrated that the appropriate averaging technique depends on geological architecture withing the volume of investigation and, in addition, the geology and petrophysics may vary away from the well bore. This implies non-stationarity and has implications for reservoir modeling. A well test can provide effective permeability for the volume of investigation, but the derivation of permeability is based is based on identification of middle time region. Middle term region reflects the infinite acting radial flow period during the pressure transient and knowledge of reservoir thickness is obtained from log interpretation. This log derived permeability is based on static reservoir conditions and may be different from effective or dynamic reservoir thickness. The early time region on the well test pressure is always affected by near well bore effects such as wellbore storage and damage due to drilling operations. This region precedes middle time region and represent the transient response in the region nearest to the well. This region is most similar to the core. Some methods proposed for as solutions to this influence are use of downhole testing tool which reduces the influence of well bore storage, and integration approach of core and production logging for the evaluation of near wellbore heterogeneity. Auguy et al., (1994) arrived at a conclusion that changing from core to the well test volume, or from early time to middle time implies statistical non-stationarity.

Given all the above issues, the comparison between well test and core permeabilities is a challenging issue and we propose a new method for analyzing porous rock properties in the laboratory by diffusion process. These processes are based on flowing and build up test of the core in the laboratory in accordance with DST well test procedure. The classical method of well test as described above is based on two dimensional radial flow (see, Warren and Price, 1961), so the present method should based on linear flow using diffusion equation and relatively long linear core. The porous core is initially saturated with air and at time $t = 0$ air is injected at constant injection rate. Pressure is measured simultaneously as pressure wave travels through the length of the sample. The measurement chart will give the pressure and time values to be used in parameter analysis. Throughout the test period it is only the inlet end that is open to admit fluid while the outlet end is closed.

Corresponding author: E-mail: david.otiede@ipsng.org, *Tel*.: +2348057443926.

If this system can be constructed, then we can solve near wellbore problem and in comparison with well test data middle time region can be easily identified.

Mathematical formulation.

$P_D(x_D,\ t_D)$ = Dimensionless pressure
P_i = initial core pressure, atmosphere
x_D = dimensionless distance from inlet end.
t_D = dimensionless time travels by the pressure wave.
K = core permeability, Darcies
$\emptyset$ = porosity, %
C = fluid compressibility, 1/atmosphere
μ = viscosity, cp.

The partial differential equation describing fluid flow in a porous medium, in terms of pressure and flow parameters, has been developed from three fundamental laws or equations. These three laws that are the basic to the solutions of all fluid flow conditions are:

1. Continuity equation that relates net mass rate inflow into the system to the rate of accumulation of mass in the system. It introduces porosity into the diffusion equation.
2. Darcy's law of flow of fluid through the rock. It relates the volumetric rate of flow to pressure and transmissibility. The transmissibility describes ability of the rock to allow flow of fluid through it.
3. Equation of state, which introduces the fluid compressibility term, defines the relationship between pressure and density.

Combining these three principles, the governing equation of flow within the porous rock is (Craft and Hawkins) mathematically given, for one dimensional flow system, as

$$\frac{\partial^2 p}{\partial x^2} = \frac{1}{\eta}\frac{\partial p}{\partial t}. \qquad (1)$$

Where $\eta = \dfrac{k}{\varphi\mu c_t}$ is called diffusivity constant, $p = p(x,t)$ is the pressure distribution as function of position and time.

The initial and boundary conditions are defined as follows:

$$P(x,0) = P_i \qquad (2)$$

The rate of injection at the injection end is expressed by Darcy's law given by

$$\left(\frac{\partial p}{\partial x}\right) = -\frac{q\mu}{kA}. \qquad (3)$$

$$p(x \to \infty, t) = p_i. \qquad (4)$$

Now by defining the following dimensionless variables we can reduce the equations to simple form and obtain a general solution in parametric form

$$p_D(x_D,t_D) = \frac{kA(p_i - p(x,t))}{q\mu L}. \qquad (5).$$

$$P_D(x_D,s) = Ae^{\sqrt{s}x_D} + Be^{-\sqrt{s}x_D} \qquad (6)$$

$$x_D = \frac{x}{L}. \qquad (6b)$$

$$t_D = \frac{\eta}{L^2}. \qquad (7)$$

Using equations (5) to (7) in equations (1) to (4) we have the transformed dimensionless equations below:

$$\frac{\partial^2 p_D}{\partial x^2_D} = \frac{\partial p_D}{\partial t_D}. \qquad (8)$$

$$p_D(x_D,0) = 0. \qquad (9)$$

$$\left(\frac{\partial p_D}{\partial x_D}\right)=1. \qquad (10)$$

$$p_D(\infty,t_D)=0. \qquad (11)$$

Method of solution

We apply Laplace transform as function of dimensionless time variable to equations eight to eleven. This linearization reduces the diffusion equation to second order ordinary differential equation in space variable. The Laplace transform is defined as

$$P_D(x_D,u)=\int_0^\infty e^{-ut}P_D(x_D,t_D)du \qquad (12)$$

where u is Laplace variable, and by application of operator (12), we have

$$\frac{d^2P}{dx_D{}^2}=uP \qquad (13)$$

And the boundary conditions are

$$\frac{P_D(x,u)}{dx_D}=\frac{1}{u} \qquad (14)$$

$$P_D[x_D\to\infty,u]=0 \qquad (15)$$

The solution to equation (13) is given as linear combination of two independent solutions

$$P_D(x_D,u)=Ae^{\sqrt{u}x_D}+Be^{-\sqrt{u}x_D} \qquad (16)$$

Using equation (15) in equation (16), we have $A=0.$, and by equation (14)
$B=\frac{1}{u^{3/2}}$. The solution in Laplace space is given by

$$P_D(x_D,u)=\frac{1}{u^{\frac{3}{2}}}Be^{-\sqrt{u}x_D} \qquad (17)$$

The inversion to time space is by the formula

$$P_D(x_D,\ t)=\int_{\alpha-i\infty}^{\alpha+i\infty}e^{ut}P_D(x_D,u)du \qquad (18)$$

We will take advantage of the property of Laplace transformation that relates the integral of a function over a finite interval with upper limit fixed, that is

$$\int_0^\infty e^{-ut}\int_0^t P_D(v)dvdu=\frac{1}{u}P_D(x_D,u) \qquad (19)$$

Since equation (17) can be separated to the form of equation (19), the inversion will be performed in two steps as follows

$$F(x_D,t_D)=\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}e^{ut}\frac{e^{-\sqrt{u}x_D}}{\sqrt{u}}du \qquad (20)$$

Here the integral has a branch point at the origin, which by analytical consideration can be transformed to one with infinite real limits. Thus, setting $\sqrt{u}=iy$ gives

$$F(x_D,t_D)=\frac{1}{\pi}\int_{-\infty}^{\infty}e^{iyx_D}e^{-t_Dy^2}du \qquad (21)$$

The factorisation of the arguments of the exponential function will make the integral separable. Thus

$$F(x_D,t_D)=\frac{1}{\pi}e^{-\frac{x_D{}^2}{4t_D}}\int_{-\infty}^{\infty}e^{-t_D\left(y-\frac{ix}{4t}\right)^2}du \qquad (22)$$

The solution to equation (22) is

$$F(x_D,t_D)=\frac{1}{\sqrt{t_D\pi}}e^{-\frac{x_D{}^2}{4t_D}} \qquad (23)$$

The dimensionless pressure is the integration of equation (23) in the limits zero to one.

$$P_D(x_D,t_D)=\frac{1}{\sqrt{\pi}}\int_0^{t_D}\frac{1}{\sqrt{z}}e^{-\frac{x_D{}^2}{4z}}dz \qquad (24)$$

The limits of integration can be transformed into semi infinite terms by making a change of variable $4z=\frac{1}{y^2}$, and performing integration by parts. That is

$$P_D(x_D,t_D)=\frac{1}{\sqrt{\pi}}\int_{\frac{1}{\sqrt{4t_D}}}^{\infty}\frac{1}{y^2}e^{-y^2x^2}dy \qquad (25)$$

$$P_D(x_D,t_D)=2\sqrt{\frac{t_D}{\pi}}\,e^{-\frac{x^2{}_D}{4t_D}}-\frac{2x^2{}_D}{\sqrt{\pi}}\int_{\frac{1}{\sqrt{4t_D}}}^{\infty}e^{-y^2x^2}dy \qquad (26)$$

The integral in equation (25) is simplified with substitution of $z=xy$ to give

$$P_D(x_D,t_D)=2\sqrt{\frac{t_D}{\pi}}\,e^{-\frac{x^2{}_D}{4t_D}}-\frac{2x_D}{\sqrt{\pi}}\int_{\frac{x_D}{\sqrt{4t_D}}}^{\infty}e^{-z^2}dz \qquad (27)$$

The integral is the well known complementary error function. The pressure distribution is therefore given by

$$P_D(x_D,t_D) = 2\sqrt{\frac{t_D}{\pi}}\; e^{-\frac{x^2{}_D}{4t_D}} - \; x_D erfc\left(\frac{x_D}{\sqrt{4t_D}}\right) \qquad (28)$$

Analysis Of Results

Equation (28) gives the pressure profile as a function of time and distance traveled by pressure wave induced by injecting unit quantity of fluid. For instant at the inlet, $x_D = 0$, the pressure profile with time is parabolic as shown in Table 1 and illustrate in Figure 1 for a flow time less than 0.01. This shows that the exponential term dominates the response throughout the flow time. In comparison with the pressure profile at 0.5 from the inlet, the exponential and error function terms cancelled out at early time and pressure profile is zero until dimensionless time of 0.019., this is shown in Figure 2 with semilog plot in Figure 3. The implication is that a semilog plot of pressure against time will give a slope of $\frac{-1}{2}$ as shown in Figure 2.

Furthermore, if the derivative of equation (28) is taken we have a diagnostic equation called derivative equations. Derivative equations are used as linearization equation for identifications of flow regimes. That is, the differential of equation (28) is

$$\frac{\partial P_D}{\partial t_D}(x_D,t_D) = \sqrt{\frac{1}{t_D\pi}}\; e^{-\frac{x^2{}_D}{4t_D}} \qquad (29)$$

Also we can use the form

$$t_D\frac{\partial P_D}{\partial t_D}(x_D,t_D) = \sqrt{\frac{t_D}{\pi}}\; e^{-\frac{x^2{}_D}{4t_D\, t_D\, t_D}} \qquad (30)$$

Equations (29) and (30) are called differential and derivative equations respectively. The generated data for these equations are shown in columns six and seven of the tables and plotted accordingly in Figures 4 and 5. It is obvious from the data in table two that diffusion pressure is an equalisation mechanism with a time lag.

Conclusion

We have demonstrated the ability of laboratory experiment to determine reservoir parameters that are comparable to the well test analysis. Since injected fluid has same characteristic as the resident fluid, the influence of capillary pressure and wettability are zero. The interpretation method will be similar to welltest as well. The interference influence can be appraised by simultaneously injecting and measuring pressure at some scaled distance from the injector end. Analysis of interference data is one method of obtaining measured areal reservoir heterogeneity and directional variations in reservoir parameters.

Table 1: pressure and derivative group with respect to time for inlet , $x_D = 0$.

Dimenless Time	erfc term	expo term	Pd	differential	Derivative
0.001	0	0.036	0.036	17.841	0.018
0.002	0	0.05	0.05	12.616	0.025
0.003	0	0.062	0.062	10.301	0.031
0.004	0	0.071	0.071	8.921	0.036
0.005	0	0.08	0.08	7.979	0.04
0.006	0	0.087	0.087	7.284	0.044
0.007	0	0.094	0.094	6.743	0.047
0.008	0	0.101	0.101	6.308	0.05
0.009	0	0.107	0.107	5.947	0.054
0.01	0	0.113	0.113	5.642	0.056
0.011	0	0.118	0.118	5.379	0.059
0.012	0	0.124	0.124	5.15	0.062
0.013	0	0.129	0.129	4.948	0.064
0.014	0	0.134	0.134	4.768	0.067
0.015	0	0.138	0.138	4.607	0.069
0.016	0	0.143	0.143	4.46	0.071
0.017	0	0.147	0.147	4.327	0.074
0.018	0	0.151	0.151	4.205	0.076
0.019	0	0.156	0.156	4.093	0.078
0.02	0	0.16	0.16	3.989	0.08

Table 2: pressure and derivative group with respect to time for inlet , $x_D = 0.5$.

Dimenless Time	erfc term	expo term	pd	differential	Derivative
0.001	0	0	0	0	0
0.002	0	0	0	0	0
0.003	0	0	0	0	0
0.004	0	0	0	0	0
0.005	0	0	0	0	0
0.006	0	0	0	0	0
0.007	0	0	0	0.001	0
0.008	0	0	0	0.003	0
0.009	0	0	0	0.006	0
0.01	0	0	0	0.011	0
0.011	0	0	0	0.018	0
0.012	0.001	0.001	0	0.028	0
0.013	0.001	0.001	0	0.04	0.001
0.014	0.001	0.002	0	0.055	0.001
0.015	0.002	0.002	0	0.071	0.001
0.016	0.003	0.003	0	0.09	0.001
0.017	0.003	0.004	0	0.11	0.002
0.018	0.004	0.005	0	0.131	0.002
0.019	0.005	0.006	0.001	0.153	0.003
0.02	0.006	0.007	0.001	0.175	0.004

Table 3: pressure data for long flow period

Dimenless Time	erfc term	expo term	pd	differential	derivative
1	0	1.128	1.128	0.564	0.564
2	0	1.596	1.596	0.399	0.798
3	0	1.954	1.954	0.326	0.977
4	0	2.257	2.257	0.282	1.128
5	0	2.523	2.523	0.252	1.262
6	0	2.764	2.764	0.23	1.382
7	0	2.985	2.985	0.213	1.493
8	0	3.192	3.192	0.199	1.596
9	0	3.385	3.385	0.188	1.693
10	0	3.568	3.568	0.178	1.784
11	0	3.742	3.742	0.17	1.871
12	0	3.909	3.909	0.163	1.954
13	0	4.068	4.068	0.156	2.034
14	0	4.222	4.222	0.151	2.111
15	0	4.37	4.37	0.146	2.185
16	0	4.514	4.514	0.141	2.257
17	0	4.652	4.652	0.137	2.326
18	0	4.787	4.787	0.133	2.394

19	0	4.918	4.918	0.129	2.459
20	0	5.046	5.046	0.126	2.523

Table 4: pressure Profile for Long flow time at $x_D = 0.5\ fom\ the\ injection\ point.$

Dimenless Time	erfc term	expo term	pd	differential	derivative
1	0.362	1.06	0.698	0.53	0.53
2	0.401	1.547	1.145	0.387	0.773
3	0.419	1.914	1.495	0.319	0.957
4	0.43	2.222	1.792	0.278	1.111
5	0.437	2.492	2.055	0.249	1.246
6	0.443	2.735	2.293	0.228	1.368
7	0.447	2.959	2.512	0.211	1.479
8	0.45	3.167	2.716	0.198	1.583
9	0.453	3.362	2.909	0.187	1.681
10	0.455	3.546	3.091	0.177	1.773
11	0.458	3.721	3.264	0.169	1.861
12	0.459	3.889	3.429	0.162	1.944
13	0.461	4.049	3.588	0.156	2.024
14	0.462	4.203	3.741	0.15	2.102
15	0.464	4.352	3.888	0.145	2.176
16	0.465	4.496	4.031	0.14	2.248
17	0.466	4.635	4.17	0.136	2.318
18	0.467	4.771	4.304	0.133	2.385
19	0.468	4.902	4.435	0.129	2.451
20	0.468	5.031	4.562	0.126	2.515

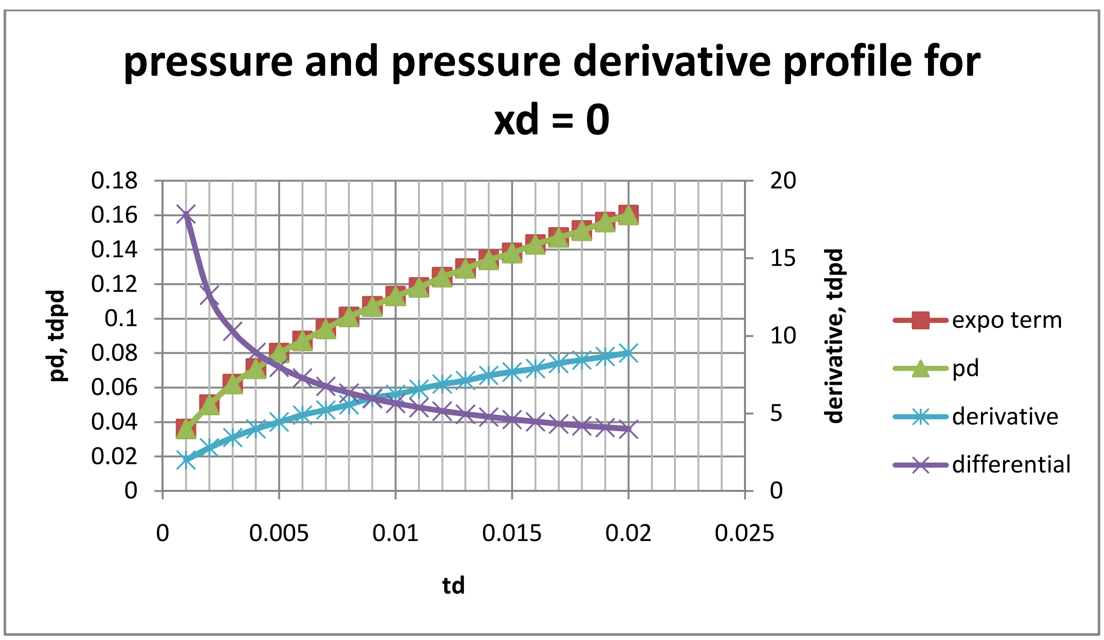

Figure 1: Dimensionless Pressure Drop profile for injection point $x_D = 0.$

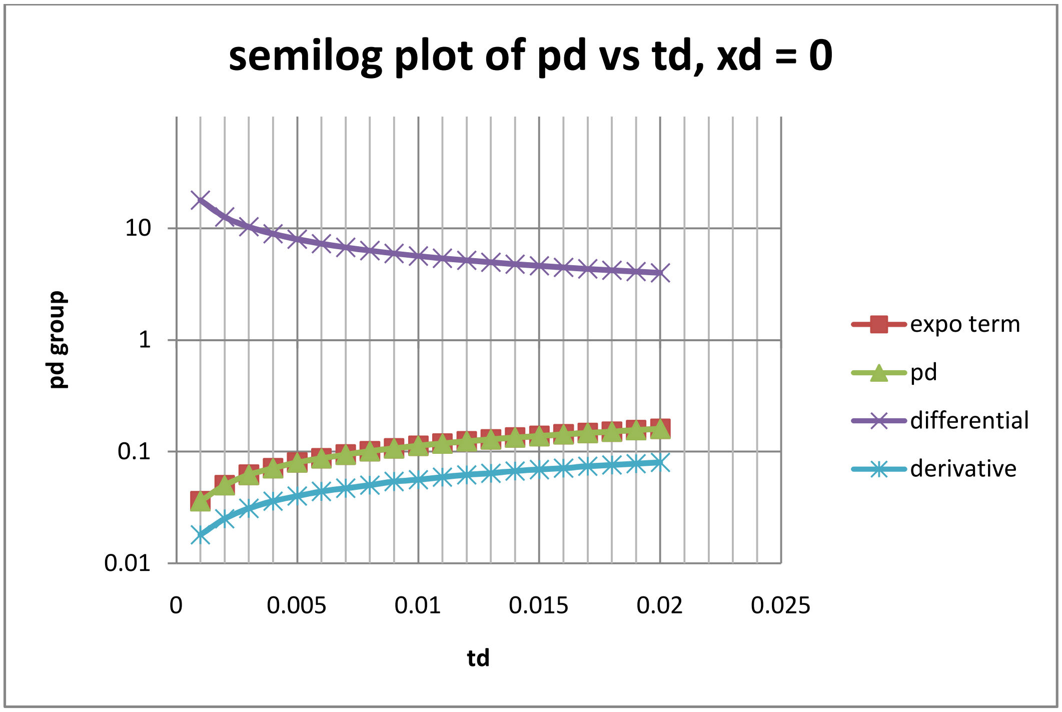

Figure 3: pressure derivative profile at $x_D = 0$.

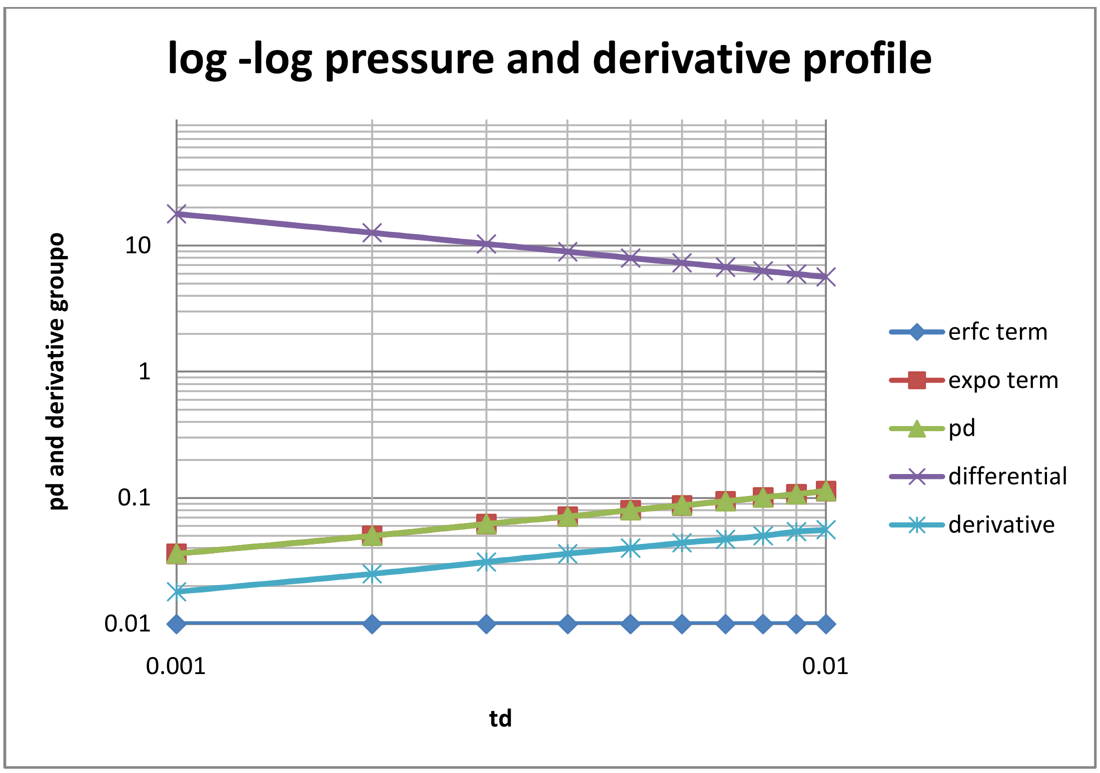

Fig. 3: Log-log plot of pressure and pressure derivative Profiles

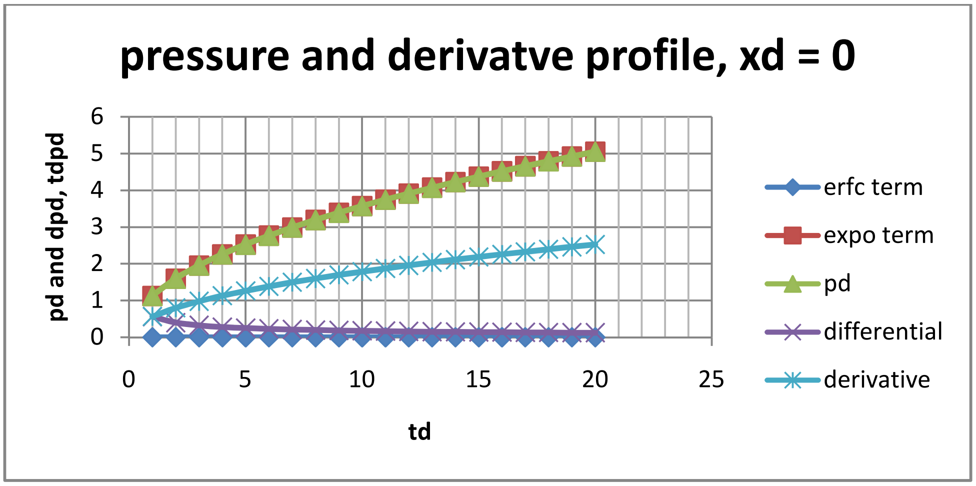

Fig. 4: Dimensionless Pressure and Derivative Profiles at $x_D = 0.0$

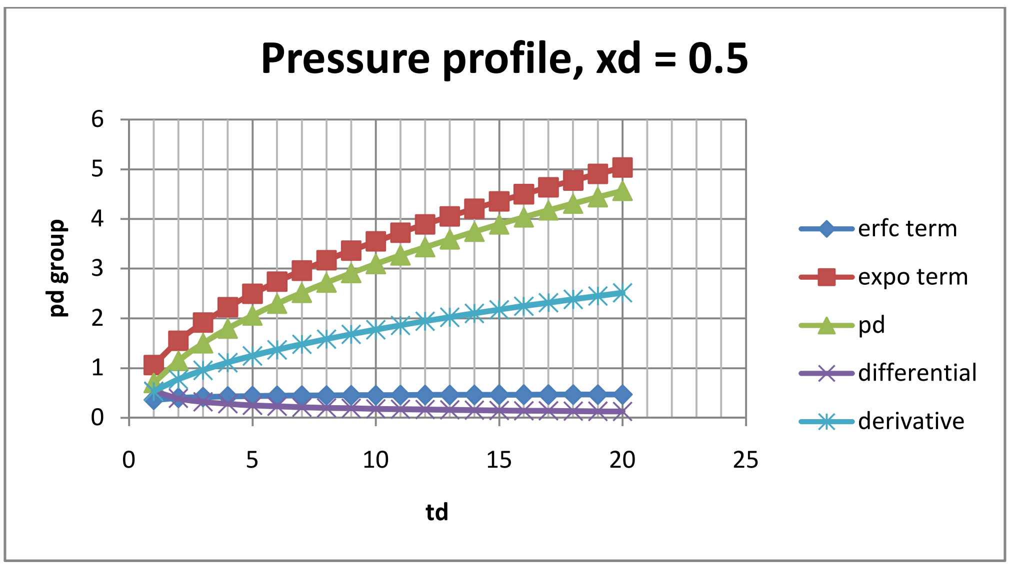

Fig. 5: Dimensionless Pressure and Derivative Profiles at $x_D = 0.5$

References

[1]. Jensen. J. L, Lake, L. W., Corbett and Goggin, D. J. : 'Statistics for Petroleum Engineers and Geoscientists' Eaglewood, Cliffs., New Jersey, Prentice Hall, 1997.

[2]. Oliver, D. S.: 'The averaging Process in Permeability estimation from well test Data', Society of Petroleum Engineers Formation Evaluation, Vol. 5, Pg. 319 – 324, 1990.

[3]. Anguy, Y., Ehlich, R; Prince, C. M.: 'the sample support problem for permeability assessment in sandstone reservoirs.', AAPG computer Applications in Geology, Pg. 37-54, 1994.

[4]. Warren, J. E. and Price, H. S,: ' Fluid flow in heterogeneous porous media', SPEJ. Sept., 1961

[5]. Craft, B. C. and Hawkins, M. F.: 'Applied Petroleum Reservoir Engineering', Prentice-Hall, Incorporations, Eaglewood Cliffs, N. j., Pg. 309-311, 1978.